Math 2B: Lesson Learning Outcomes

LESSONS 0: THE SIX MAJOR PROBLEMS OF LINEAR ALGEBRA

- ☐ State the applied math modeling problem.
  - Explain the Applied Math Modeling Process.
  - Explain where matrices and vectors arise in this process.
  - Explain why Jeff is head-over-heels in love with Applied Linear Algebra. In other words, explain why Jeff says that applied linear algebra is so powerful in post-1945 knowledge economy. Why is this class not call nonlinear algebra? How does the adjective “linear” relate to the development of computers.
  - Explain where Applied Linear Algebra shows up in Jeff’s Concept Map of Applied Math in Action.
- ☐ State the matrix-vector multiplication problem (MVMP).
  - Identify the known and unknown information in the MVMP.
  - Specify the dimensions for all matrices in the MVMP.
  - Identify the domain and codomain of the MVMP.
- ☐ State the nonsingular linear-systems problem (NLSP).
  - Identify the known and unknown information in the NLSP.
  - Specify the dimensions for all matrices in the NLSP.
  - Identify the domain and codomain of the NLSP.
- ☐ State the general linear-systems problem (GLSP).
  - Identify the known and unknown information in the GLSP.
  - Specify the dimensions for all matrices in the GLSP.
  - Identify the domain and codomain of the GLSP.
- ☐ State the full-rank least-squares problem (FRLSP).
  - Identify the known and unknown information in the FRLSP.
  - Specify the dimensions for all matrices in the FRLSP.
  - Identify the domain and codomain of the FRLSP.
- ☐ State the standard eigenvalue problem (SEP).
  - Identify the known and unknown information in the SEP.
  - Specify the dimensions for all matrices in the SEP.
  - Identify the domain and codomain of the SEP.

LESSONS 1: INTRODUCTION TO SET THEORY

- ☐ Given sets A and B, prove $A \subseteq B$
- ☐ Given sets A and B, prove A = B
- ☐ Describe sets using either element enumeration or set builder notation
- ☐ Identify Important Number Systems using proper notation

LESSONS 2: RELATIONS AND FUNCTIONS

- ☐ Identify Domain Space, Domain, Codomain and Range of a given relation
- ☐ Convert natural numbers into binary representations
- ☐ Convert binary numbers into natural numbers
- ☐ Properly use set theory definition of function
- ☐ Apply the gray scale function for storing shades of gray in a computer

LESSONS 3: VECTORS AND MODELING

- ☐ Use column vectors to create a vertex model of points in $\mathbb{R}^2$
- ☐ Use column vectors to describe data from Hooke’s law experiment
- ☐ Use column vectors to model Ohm’s law experiment
- ☐ Apply Ohm’s Law to describe relations between voltage and current in circuit.
- ☐ Use column vectors to capture position data for mass-spring chain
- ☐ Discretize a given function $y = f(x)$ to produce input vector $\vec{\boldsymbol{x}}$ and output vector $\vec{\boldsymbol{y}}$
- ☐ Properly identify equality of vectors
- ☐ Use row vectors appropriately

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LESSONS 4: VECTOR ARITHMETIC

- ☐ Use scalar-vector multiplication and vector addition to model for Hooke's law
- ☐ Use vector-vector addition to create the displacement vector for a mass-spring system with *n* masses and *(n+1)* springs where n = 2, 3, 4, 5
- ☐ Properly apply Algebraic Properties of vector addition and scalar multiplication
- ☐ Properly apply Algebraic Properties of vector transposes

LESSONS 5: INNER PRODUCTS AND VECTOR NORMS

- ☐ Use inner product to calculate Riemann sums (NOT WINTER 2017)
- ☐ Use inner products to calculate your final grade in Math 2B
- ☐ Use inner products to calculate the voltage across a circuit element
- ☐ Use inner products to write Kirchoff's current law for any node of an ideal circuit
- ☐ Properly identify and apply Algebraic Properties inner products
- ☐ Prove any of the algebraic properties of inner products
- ☐ Prove the Pythagorean Theorem and the Law of Cosines
- ☐ Prove the cosine formula for dot products
- ☐ Properly apply Algebraic Properties of 2-norm of a vector
- ☐ Prove the algebraic properties of the 2-norm of a vector
- ☐ Properly identify and apply definition of Orthogonality between vectors

LESSONS 6: LINEAR COMBINATIONS, SPANS AND LINEAR INDEPENDENCE

- ☐ Find the span of a given set of vectors
- ☐ Identify linearly independent vectors
- ☐ Identify linearly dependent vectors
- ☐ Discuss properties of span of given set of vectors

LESSONS 7: MATRICES AND MODELING

- ☐ Create the incidence matrix for a given undirected graph
- ☐ Create incidence matrix for a given directed graph
- ☐ Identify and use the 2D wireframe model for given polygon
- ☐ Set up matrix model for a given mass-spring chain with *n* masses and *(n+1)* springs
- ☐ Properly use the entry operator to refer to an entry of a matrix
- ☐ Properly identify and apply matrix model for digital image
- ☐ Identify dimensions of a given matrices
- ☐ Properly use definition of equal matrices

LESSON 8: ANATOMY OF MATRICES

- ☐ Discuss the properties of the matrix-vector multiplication problem in detail
- ☐ Identify and apply colon notation to denote the columns of a matrix.
- ☐ Identify and apply colon notation to denote the rows of a matrix.
- ☐ Properly use colon notation for row and column partition of a matrix

LESSONS 9: OUTER PRODUCTS AND MATRIX ARITHMETIC

- ☐ Identify the definition of the outer product
- ☐ Calculate the outer product between two given vectors
- ☐ Use outer products to calculate matrix units
- ☐ Recognize the difference between an outer and inner product by identifying dimensions
- ☐ Use outer products and matrix arithmetic to calculate any of the three elementary matrices
    - Shear Matrices: $S_{ik}(c)$
    - Dilation Matrices: $D_j(c)$
    - Permutation (Transposition) Matrices: P_{ik}
- ☐ Apply definitions of elementary matrices $S_{ik}(c)$, $D_j(c)$, P_{ik} to do matrix arithmetic
- ☐ Apply the Algebraic Properties of sum, difference and scalar multiple of a matrix
- ☐ Properly identify and apply algebraic properties of transpose of a matrix

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LESSON 10: THE MATRIX-VECTOR MULTIPLICATION PROBLEM (MVMP)

- ☐ Discuss the properties of the matrix-vector multiplication problem in detail
  - What is given and what is unknown?
  - How does this relate to functions (think domain and codomain)?
  - How is this related to the square linear-systems problem?
  - What does the matrix-vector multiplication problem have to do with linear combinations?
- ☐ Identify and apply column-partition version of matrix-vector multiplication
- ☐ Identify and apply the entry-by-entry version of matrix-vector multiplication
- ☐ Use matrix-vector multiplication to analyze mass-spring chains
- ☐ Use matrix-vector multiplication to calculate voltage drops across ideal circuit elements
- ☐ Use matrix-vector multiplication to state KCL at all nodes of a circuit
- ☐ Use matrix-vector multiplication to state Ohm's Law for all resistors in a circuit
- ☐ Create Vandermonde matrix to sample a given nth degree polynomial

LESSON 11: MATRIX-MATRIX MULTIPLICATION

- ☐ Determine if two matrices are conformable for matrix multiplication
- ☐ Properly identify the dimensions of a matrix-matrix product
- ☐ Identify the dimensions of left and right arguments of matrix-matrix multiplication
- ☐ Identify left and right arguments of a matrix-matrix multiplication
- ☐ Use column-partition version of matrix-matrix multiplication to multiply a matrix on the right
- ☐ Use row-partition version of matrix-matrix multiplication to multiply a matrix on the left
- ☐ Use entry-by-entry version of matrix-matrix multiplication to quickly calculate individual entries of a matrix-matrix product
- ☐ Prove that all forms of matrix-matrix multiplication are equivalent.
  - Prove matrix-matrix multiplication by column is equal to matrix-matrix multiplication by row.
  - Prove matrix-matrix multiplication by column is equal to matrix-matrix multiplication by entry.
  - Prove matrix-matrix multiplication by row is equal to matrix-matrix multiplication by entry.
- ☐ Use row-partition version of matrix-matrix multiplication multiply a matrix on the left by:
  - Shear matrix: $S_{ik}(c)$
  - Dilation matrix: $D_j(c)$
  - Permutation matrix: P_{ik}

LESSON 12: THE NONSINGULAR LINEAR-SYSTEMS PROBLEM (NLSP)

- ☐ Recall the two fundamental questions about linear-systems problem
  - The existence problem
  - The uniqueness problem
- ☐ Define the nonsingular linear-system problem and discuss
  - What is given and what is unknown?
  - How does this relate to functions (think domain and codomain)?
  - How is this related to the matrix-vector multiplication problem?
  - How many solutions can exist to this problem?
  - How is the nonsingular linear-systems problem related to the span of the columns of matrix A?
- ☐ Set up and solve a nonsingular linear-systems problem for a given mass-spring chain with n masses and $(n+1)$ springs where $n = 2, 3, 4, 5, 6$
- ☐ Set up linear systems problem for linear spline interpolation
  - See examples 5.1.1 & 5.1.2 in Lesson 12 Notes
- ☐ Set up a linear systems problems using a Vandermonde matrix for polynomial modeling.
  - Quadratic Polynomial: See example 5.1.4 in Jeff's Lesson 12 Notes
  - Linear Polynomial: See example 5.1.5 in Jeff's Lesson 12 Notes
- ☐ For a given diagonal matrix $D \in \mathbb{R}^{n\times n}$ with nonzero diagonal elements, solve the nonsingular linear-systems problem $D \cdot \boldsymbol{x} = \boldsymbol{b}$
- ☐ For a given upper-triangular matrix $U \in \mathbb{R}^{n\times n}$ with nonzero diagonal elements, properly apply backward substitution algorithm to solve the nonsingular linear-systems problem $U \cdot \boldsymbol{x} = \boldsymbol{y}$
- ☐ For a given lower-triangular matrix $L \in \mathbb{R}^{n\times n}$ with nonzero diagonal elements, properly apply forward substitution algorithm to solve the nonsingular linear-systems problem $L \cdot \boldsymbol{y} = \boldsymbol{b}$
- ☐ Multiply by elementary matrices to create an upper-triangular matrix U
- ☐ Apply matrix-matrix multiplication to solve linear systems problem

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LESSON 13: MATRIX INVERSES

- ☐ Recall and apply the definition of the inverse of a square matrix
- ☐ State, apply and derive the inverse formulas for elementary matrices
  - Shear matrices: $(S_{ik}(c))^{-1} = S_{ik}(-c)$
  - Dilation matrices: $\left(D_j(c) \right)^{-1} = D_j\left(\frac{1}{c}\right)$
  - Permutation matrices: $(P_{ik})^{-1} = {P_{ik}}^T = P_{ik}$
- ☐ State and prove the following properties of Matrix inverses:
  - Prove $(AB)^{-1} = B^{-1}A^{-1}$ AND $(A^T)^{-1} = (A^{-1})^T$
- ☐ Find the inverse of a 2-by-2 matrix using formula
- ☐ Use elementary row operations matrices to generate the inverse of an n-by-n matrix A
- ☐ Use a given matrix inverse A^{-1} to find solution to linear system $A\,\boldsymbol{x} = \boldsymbol{b}$
- ☐ Use elementary matrices to derive Cramer's Rule for the inverse of a 2-by-2 matrix, given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

LESSON 14: THE INVERTIBLE MATRIX THEOREM (IMT)

- ☐ Properly identify and apply parts 1 – 22 of the Invertible Matrix Theorem

LESSON 15: LU FACTORIZATION WITHOUT PIVOTING

- ☐ Find the LU Factorization of a given 3-by-3 or 4-by-4 matrix
- ☐ Use a given LU Factorization $A = LU$ to solve given linear system problem $A \cdot \boldsymbol{x} = \boldsymbol{b}$
- ☐ Use elementary row operations matrices to generate LU factorization
- ☐ Prove that the product of lower-triangular matrices is lower-triangular (or upper-triangular case)

LESSON 16: DETERMINANTS

- ☐ Identify and apply the definition of the determinant for 2-by-2 matrices: $A \in \mathbb{R}^{2\times 2}$
- ☐ Identify and apply the definition of the determinant for 3-by-3 matrices: $A \in \mathbb{R}^{3\times 3}$
- ☐ Identify and apply the definition of the determinant for an upper-triangular matrix: $U \in \mathbb{R}^{n\times n}$
- ☐ Apply properties of determinants to analyze linear system problem
- ☐ Identify and apply properties 1 – 9 of determinants
- ☐ Given $A \in \mathbb{R}^{n\times n}$, use your calculator to find the determinant: $\det(A)$
- ☐ Properly identify and use permutation definition of determinants

LESSON 17: THE GENERAL LINEAR-SYSTEMS PROBLEM (GLSP)

- ☐ Define the general linear-systems problem (GLSP) and discuss
  - What is given and what is unknown?
  - How does this relate to functions (think domain and codomain)?
  - How is this related to the matrix-vector multiplication problem?
  - What do solutions of the general linear-systems problem have to do with the span of the columns of matrix A?
- ☐ Given an m-by-n matrix, transform into REF or RREF using multiplication by elementary matrices
- ☐ Given an m-by-n matrix A, use $\text{RREF}(A)$to identify linearly independent columns
- ☐ If $U = \text{RREF}(A)$, prove $A \cdot \boldsymbol{x} = \boldsymbol{0}$ if and only if $U \cdot \boldsymbol{x} = \boldsymbol{0}$
- ☐ Use your calculator to transform a given matrix A into $U = \text{RREF}(A)$

LESSON 18: SOLUTIONS SETS TO THE GENERAL LINEAR-SYSTEMS PROBLEM

- ☐ For a given matrix $A \in \mathbb{R}^{m\times n}$, define the homogeneous linear-systems problem
- ☐ Find all solutions to linear systems problem $A \cdot \boldsymbol{x} = \boldsymbol{b}$ using the equivalent linear-systems problem $U \cdot \boldsymbol{x} = \boldsymbol{y}$, where
- ☐ $U = E \cdot A = \text{RREF(A)}$, $\boldsymbol{y} = E \cdot \boldsymbol{b}$, AND $E = E_t \cdots E_2 \cdot E_1$ is a product of t elementary matrices
- ☐ Given $A \in \mathbb{R}^{m\times n}$, use $\text{RREF}(A)$ to solve homogeneous linear system: $A \cdot \boldsymbol{x} = \boldsymbol{0}$
- ☐ Discuss how the solution set for a GLSP relates to the the superposition principle of matrix-vector multiplication given by $\boldsymbol{A} \cdot (c_1\boldsymbol{x_1} + c_2\boldsymbol{x_2}) = c_1(\boldsymbol{A} \cdot \boldsymbol{x_1}) + c_2(\boldsymbol{A} \cdot \boldsymbol{x_2})$
- ☐ Properly construct the solution set to the general linear-systems problem $A \cdot \boldsymbol{x} = \boldsymbol{b}$ using the formula

$$\boldsymbol{x} = \boldsymbol{x}^* + c_1 \cdot \boldsymbol{z_1} + \cdots + c_d \cdot \boldsymbol{z_d}$$

where $\boldsymbol{x}^* \in \mathbb{R}^{\boldsymbol{n}}$ is a particular solution to our original GLSP, d is the number of nonpivot columns of $A \in \mathbb{R}^{m\times n}$, and $\boldsymbol{z_1}, \dots, \boldsymbol{z_d} \in \mathbb{R}^n$ are linearly independent solution to our associated homogeneous linear-systems problem.

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LESSON 19: VECTOR SPACES

- ☐ Properly identify algebraic definition of vector spaces
  - Prove that $\mathbb{R}^n$ is a vector space
  - Prove that $\mathbb{R}^{m\times n}$ is a vector space
- ☐ Use properties of vector spaces to identify vector spaces
  - Give examples of subspaces of $\mathbb{R}^n$
  - Give examples, with reasoning, of subsets of $\mathbb{R}^n$ that are not subspaces
- ☐ Prove that a subset W of vector space V is a subspace

LESSON 20: NULL AND COLUMN SPACES

- ☐ For $A \in \mathbb{R}^{m\times n}$, prove each of the following
  - $\text{Null}(A)$ is a subspace of $\mathbb{R}^n$
  - $\text{Col}(A^T)$ is a subspace of $\mathbb{R}^n$
  - $\text{Null}(A^T)$ is a subspace of $\mathbb{R}^m$
  - $\text{Col}(A)$ is a subspace of $\mathbb{R}^m$
- ☐ Given m-by-n matrix A, show how to use $\text{RREF}(A)$ to find $\text{Null}(A)$ and the $\text{Col}(A)$
- ☐ Discuss how to use $\text{Col}(A)$ to answer the existence problem for a GLSP
- ☐ Discuss how to use the $\text{Null}(A)$ to answer the uniqueness problem for a GLSP

LESSON 21: DIMENSION AND RANK

- ☐ Recall and apply the definition of basis vectors.
- ☐ Given a set of vectors $\{\boldsymbol{a_k}\}_{k=1}^n \subseteq \mathbb{R}^m$, find a basis for the span of this set.
- ☐ Given a set of vectors $\{\boldsymbol{a_k}\}_{k=1}^n \subseteq \mathbb{R}^m$, find the dimension of $\text{Span}(\{\boldsymbol{a_k}\}_{k=1}^n)$
- ☐ Given $A \in \mathbb{R}^{m\times n}$, find $\text{rank}(A) = \dim\big(\text{Col}(A)\big)$
- ☐ Given $A \in \mathbb{R}^{m\times n}$, find $\dim\big(\text{Null}(A)\big)$
- ☐ Given a list of vectors $\{\boldsymbol{a_k}\}_{k=1}^n$, find the dimension of $\text{Span}(\{\boldsymbol{a_k}\}_{k=1}^n)$
- ☐ Given $A \in \mathbb{R}^{m\times n}$, discuss how the $\dim\big(\text{Null}(A)\big)$ relates to the number of solution of the general linear-systems problem and the homogeneous linear-systems problem.

LESSON 22: INTRODUCTION TO THE LEAST-SQUARES PROBLEM

- ☐ Set up least-square problem using list of data $\{(x_i, y_i)\}_{i=1}^m$ and Vandermonde matrix
  - Use least-squares problem to set up linear model
  - Use least-squares problem to set up quadratic model
- ☐ Solve normal equations to find least-squares solution
- ☐ Discuss connection between rank(A) and least squares solution
- ☐ Use least-squares model to interpolate or extrapolate values from data set

LESSON 23: ORTHOGONAL SETS

- ☐ Prove that Nul(A) is orthogonal to Col(A^T)
- ☐ Prove that Nul(A^T) is orthogonal to Col(A)
- ☐ Derive the orthogonal projection formula used in the Gram Schmidt processes

LESSON 24: ORTHOGONAL PROJECTIONS

- ☐ Given two vectors $\boldsymbol{b}, \boldsymbol{y} \in \mathbb{R}^{\boldsymbol{n}}$, find $\text{Proj}_Y(\boldsymbol{b})$ where $Y = \text{Span}(\boldsymbol{y})$
- ☐ Given two vectors $\boldsymbol{b}, \boldsymbol{y} \in \mathbb{R}^{\boldsymbol{n}}$, find $\text{Proj}_{Y^\perp}(\boldsymbol{b})$ where $Y^\perp = [\text{Span}(\boldsymbol{y})]^\perp$
- ☐ Explain the origins of algebra behind orthogonal projections using dot-product

LESSON 25: THE CLASSICAL GRAM SCHMIDT ALGORITHM

- ☐ Given matrix A, find an orthogonal basis for Col(A) using Classical Gram-Schmidt
- ☐ Use Classical Gram-Schmidt to solve a least-squares problem

LESSON 26: QR FACTORIZATION VIA MODIFIED GRAM SCHMIDT ALGORITHM

- ☐ Given a matrix A, produce the QR factorization of A using Modified Gram Schmidt
- ☐ Use the QR Factorization of a matrix to solve a least-squares problem

- ☐ Compare and contrast Classical Gram Schmidt and Modified Gram Schmidt

LESSON 27: THE STANDARD EIGENVALUE PROBLEM

- ☐ Starting from the matrix-version of the differential equation for undamped simple harmonic oscillators given by $M\ddot{\boldsymbol{u}} + K\boldsymbol{u} = \boldsymbol{0}$, derive the statement of the eigenvalue problem $A\boldsymbol{x} = \lambda\,\boldsymbol{x}$ associated with this differential equation (use ansatz).
- ☐ Given a 2-by-2 matrix A, find the eigenvalues and eigenvectors of A
- ☐ Given a 3-by-3 matrix A, find the eigenvalues and eigenvectors of A

LESSON 28: THE CHARACTERISITC EQUATION

- ☐ Given a 2-by-2 or 3-by-3 matrix A, find its characteristic polynomial
- ☐ Factor a characteristic polynomial to find its roots
- ☐ Prove that similar matrices have identical characteristic polynomials

LESSON 29: DIAGONALIZATION

- ☐ Diagonalize a given 2-by-2 or 3-by-3 matrix A
- ☐ Identify the geometric and algebraic multiplicity of given eigenvalues