Jeff Anderson's Math 1C Calculus III
Content-Specific Learning Objectives

Multivariable Differentiation

LESSONS 0: THE LEAST-SQUARES PROBLEM

- ☐ Discuss the major themes of part 1 of Math 1C:
  - ☐ State the limit definition of the ordinary derivative from Math 1A.
  - ☐ State the second derivative test from Math 1A.
  - ☐ Given a single variable function, find the maximum and minimum values of this function using the second derivative test from Math 1A.
  - ☐ Describe the forward and backward problems of single-variable differentiation. Make explicit connections to the work you did in Math 1AB.
  - ☐ Describe the forward and backward problems of multivariable-variable differentiation. Make explicit connections to the work you will do in Math 1CD.
  - ☐ Compare and contrast single-variable differentiation with multivariable differentiation.
  - ☐ Construct a linear least-squares problem using a given data set. Make sure you can
    - Set up a model for the error e_i between the ith data point (x_i, y_i) and any associated linear model $y = f(x) = mx + b$.
    - Using the model for the errors, set up the least-squares problem for this input data. In particular, create a two variable function $E_2(b, m)$ that you can use to solve create the ``best-fit" model for this data. Explain in detail the choices that you made to construct your error function E_2. Why do you use the total squared error (and not the sum of the individual errors e_i or the sum of the absolute value of the individual errors $|e_i|$)?
    - Explain how we will use multivariable calculus to find the line of best fit. Make explicit connections with the second derivative test you learned in Math 1A.

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LESSONS 1: VECTORS IN $\mathbb{R}^2$

- ☐ State the definition of the points in $\mathbb{R}^2$
  - ☐ Properly use point notation $P(x, y)$ for points in $\mathbb{R}^2$
  - ☐ Identify first and second coordinates of a point $P(x, y)$ in $\mathbb{R}^3$
  - ☐ Graph points on the 2D Cartesian plane
- ☐ State the definition of vectors in $\mathbb{R}^2$
  - ☐ Properly use vector notation $\vec{\boldsymbol{x}} = < x,\ y\ >$ for vectors in $\mathbb{R}^2$
  - ☐ Identify initial point (tail) and terminal point (head) of vector $\vec{\boldsymbol{x}} = \overrightarrow{P_1P_2}$ in $\mathbb{R}^2$
  - ☐ Graph vectors on the 2D Cartesian plane
- ☐ Derive formula for the two-norm (magnitude) of a vector $\vec{\boldsymbol{x}} = \overrightarrow{P_1P_2}$ in $\mathbb{R}^2$
  - ☐ For the 2-norm, explain the function map notation: $\|\ \cdot\ \|_{\mathbf{2}}\ :\ \mathbb{R}^2 \longrightarrow \mathbb{R}$
  - ☐ Properly use two-norm notation $\|\ \vec{\boldsymbol{x}}\ \|_{\mathbf{2}}$ for vectors in $\mathbb{R}^2$
  - ☐ Explain how the formula for the two-norm relates to the Pythagorean theorem
  - ☐ Calculate the two norm of a given vector in $\mathbb{R}^2$
  - ☐ Show that for any $c \in \mathbb{R}$ and $\vec{\boldsymbol{x}} \in \mathbb{R}^2$, we have $\|\ c\ \cdot\ \vec{\boldsymbol{x}}\ \|_{\mathbf{2}} = |c|\ \cdot\ \|\ \vec{\boldsymbol{x}}\ \|_{\mathbf{2}}$
- ☐ State and apply the definition of vector-vector addition in $\mathbb{R}^2$
  - ☐ For vector-vector addition, explain the function map notation: $+ : \mathbb{R}^2\ \times\ \mathbb{R}^2 \longrightarrow \mathbb{R}^2$
  - ☐ Add two given vectors together in $\mathbb{R}^2$
  - ☐ Interpret vector-vector addition geometrically via the triangle and parallelogram laws
- ☐ State and apply the definition of scalar-vector multiplication in $\mathbb{R}^2$
  - ☐ For scalar-vector multiplication, explain the function map notation: $\cdot : \mathbb{R}\ \times\ \mathbb{R}^2 \longrightarrow \mathbb{R}^2$
  - ☐ Interpret scalar-vector multiplication geometrically
  - ☐ Identify what it means for two vectors to be in the "same direction"
  - ☐ Calculate a scalar-vector product in $\mathbb{R}^2$
  - ☐ Given a vector $\vec{\boldsymbol{x}}$ in $\mathbb{R}^2$, use scalar-vector product to create a unit vector in the direction of $\vec{\boldsymbol{x}}$

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LESSONS 2: VECTORS IN $\mathbb{R}^3$

- ☐ State the definition of the points in $\mathbb{R}^3$
    - ☐ Properly use point notation $P(x, y, z)$ for points in $\mathbb{R}^3$
    - ☐ Identify first, second, and third coordinates of a point $P(x, y, z)$in $\mathbb{R}^3$
    - ☐ Graph points on the 3D Cartesian plane (hard to do by hand, better in Mathematica)
- ☐ State the equations for the Coordinate Planes in $\mathbb{R}^3$
    - ☐ State the equation for the xy-plane
    - ☐ State the equation for the xz-plane
    - ☐ State the equation for the yz-plane
    - ☐ Explain why each of these equations correspond to each plane (in terms of ordered triplets)
- ☐ State the definition of vectors in $\mathbb{R}^3$
    - ☐ Properly use vector notation $\vec{\boldsymbol{x}} = < x,\ y,\ z\ >$ for vectors in $\mathbb{R}^3$
    - ☐ Identify initial point (tail) and terminal point (head) of vector $\vec{\boldsymbol{x}} = \overrightarrow{P_1P_2}$ in $\mathbb{R}^3$
    - ☐ Graph vectors on the 3D Cartesian plane (hard to do by hand, better in Mathematica)
- ☐ Derive formula for the two-norm (magnitude) of a vector $\vec{\boldsymbol{x}} = \overrightarrow{P_1P_2}$ in $\mathbb{R}^3$
    - ☐ For the 2-norm, explain the function map notation: $\|\cdot\|_{\mathbf{2}} : \mathbb{R}^3 \longrightarrow \mathbb{R}$
    - ☐ Properly use two-norm notation $\| \vec{\boldsymbol{x}} \|_{\mathbf{2}}$ for vectors in $\mathbb{R}^3$
    - ☐ Explain how the formula for the two-norm relates to the Pythagorean theorem
    - ☐ Calculate the two norm of a given vector in $\mathbb{R}^3$
    - ☐ Show that for any $c \in \mathbb{R}$ and $\vec{\boldsymbol{x}} \in \mathbb{R}^3$, we have $\| c \cdot \vec{\boldsymbol{x}} \|_{\mathbf{2}} = |c| \cdot \| \vec{\boldsymbol{x}} \|_{\mathbf{2}}$
- ☐ State and apply the definition of vector-vector addition in $\mathbb{R}^3$
    - ☐ For vector-vector addition, explain the function map notation: $+ : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$
    - ☐ Add two given vectors together in $\mathbb{R}^3$
- ☐ State and apply the definition of scalar-vector multiplication in $\mathbb{R}^3$
    - ☐ For scalar-vector multiplication, explain the function map notation: $\cdot : \mathbb{R} \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$
    - ☐ Interpret scalar-vector multiplication geometrically
    - ☐ Identify what it means for two vectors to be in the "same direction"
    - ☐ Calculate a scalar-vector product in $\mathbb{R}^3$
    - ☐ Given a vector $\vec{\boldsymbol{x}}$ in $\mathbb{R}^3$, use scalar-vector product to create a unit vector in the direction of $\vec{\boldsymbol{x}}$
- ☐ State the equation for a sphere in $\mathbb{R}^3$
    - ☐ Given an equation for a sphere, complete the square to find the center and radius of sphere

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LESSONS 3: THE DOT PRODUCT IN $\mathbb{R}^n$

- ☐ For the dot product in $\mathbb{R}^2$, explain the function map notation: $\cdot : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$
- ☐ For the dot product in $\mathbb{R}^3$, explain the function map notation: $\cdot : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$
- ☐ State and apply the component form of the dot product between vectors in $\mathbb{R}^n$ $(n = 2, 3)$
- ☐ Given two vectors, calculate their dot product using the component form of the dot product
- ☐ State, prove and apply the algebraic properties of the dot product
  - ☐ Length via dot product: $\| \vec{\boldsymbol{x}} \|_2^2 = \vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{x}}$
  - ☐ Symmetry of dot product: $\vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{y}} = \vec{\boldsymbol{y}} \cdot \vec{\boldsymbol{x}}$
  - ☐ Bi-linearity of the Dot Product
    - ☐ Linearity in Left Argument: $(\vec{\boldsymbol{x}} + \vec{\boldsymbol{y}}) \cdot \vec{\boldsymbol{z}} = \vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{z}} + \vec{\boldsymbol{y}} \cdot \vec{\boldsymbol{z}}$
    - ☐ Linearity in Right Argument: $\vec{\boldsymbol{x}} \cdot (\vec{\boldsymbol{y}} + \vec{\boldsymbol{z}}) = \vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{y}} + \vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{z}}$
  - ☐ Homogeneity of the Dot Product: $(\alpha \vec{\boldsymbol{x}}) \cdot \vec{\boldsymbol{y}} = \vec{\boldsymbol{x}} \cdot (\alpha \vec{\boldsymbol{y}}) = \alpha(\vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{y}})$
  - ☐ Dot product with zero: $\vec{\boldsymbol{0}} \cdot \vec{\boldsymbol{x}} = \vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{0}} = 0$
- ☐ State, prove and apply the cosine formula for the dot product
  - ☐ State and prove the Pythagorean theorem
  - ☐ State and prove the Law of Cosines (Accute and Obtuse cases)
  - ☐ State and prove the cosine formula for the dot product
- ☐ State, prove and apply the orthogonality theorem for dot product: $\vec{\boldsymbol{x}} \perp \vec{\boldsymbol{y}} \iff \vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{y}} = 0$
- ☐ Derive the projection formula for $\text{Proj}_{\vec{\boldsymbol{x}}}(\vec{\boldsymbol{y}})$
  - ☐ Draw the visual associated with the project problem and explain the geometric significance of projection vector $\vec{\boldsymbol{p}}$ and residual vector $\vec{\boldsymbol{r}}$
  - ☐ Explain how we use the Orthogonality theorem of dot product to construct our projection
  - ☐ Using derivation, create the formula for the projection function $\text{Proj}_{\vec{\boldsymbol{x}}}(\vec{\boldsymbol{y}})$
  - ☐ Explain why the scalar project $\text{Scal}_{\vec{\boldsymbol{x}}}(\vec{\boldsymbol{y}})$ gives the length of vector $\vec{\boldsymbol{p}} = \text{Proj}_{\vec{\boldsymbol{x}}}(\vec{\boldsymbol{y}})$
  - ☐ Given two vectors, calculate $\text{Proj}_{\vec{\boldsymbol{x}}}(\vec{\boldsymbol{y}})$ and $\text{Scal}_{\vec{\boldsymbol{x}}}(\vec{\boldsymbol{y}})$
- ☐ Discuss the two important utilities of the cross products product
  - ☐ Explain how the dot product is used as a measurement of parallelity
  - ☐ Explain how the dot product is used to project one vector onto the direction of another vector

LESSONS 4: THE CROSS PRODUCT IN $\mathbb{R}^3$

- ☐ For the cross product in $\mathbb{R}^3$, explain the function map notation: $\times : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$
- ☐ Given two vectors $\vec{\boldsymbol{x}} = < x_1,\ y_1 >$ and $\vec{\boldsymbol{y}} = < x_2,\ y_2 >$, explain how to find the area of the parallelogram formed by the vectors $\vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{y}}$
  - ☐ Derive the component form of the area: $x_1\, y_2 - x_2\, y_1$
  - ☐ Derive the sine form of the area: $\| \vec{\boldsymbol{x}} \|_2 \, \| \vec{\boldsymbol{y}} \|_2 \, \text{Sin}(\, \theta\,)$ where θ is the angle between $\vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{y}}$
  - ☐ Explain how the component form for the area of a parallelogram is a proxy measurement for perpendicularity (how is $x_1\, y_2 - x_2\, y_1$ related to angle θ between $\vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{y}}$)
- ☐ Explain how to use the right-hand rule to find each of the following cross products:
  - ☐ $\vec{\boldsymbol{\imath}} \times \vec{\boldsymbol{\jmath}}$ and $\vec{\boldsymbol{\jmath}} \times \vec{\boldsymbol{\imath}}$
  - ☐ $\vec{\boldsymbol{\imath}} \times \vec{\boldsymbol{k}}$ and $\vec{\boldsymbol{k}} \times \vec{\boldsymbol{\imath}}$
  - ☐ $\vec{\boldsymbol{\jmath}} \times \vec{\boldsymbol{k}}$ and $\vec{\boldsymbol{k}} \times \vec{\boldsymbol{\jmath}}$
- ☐ Explain why we use the right-hand rule to define the direction of a cross product
- ☐ Explain where the component form of the cross product between vectors in $\mathbb{R}^3$ comes from
- ☐ State and apply the component form of the cross product between vectors in $\mathbb{R}^3$
- ☐ Given two vectors, calculate their cross product using the component form
- ☐ State, prove and apply the algebraic properties of the dot product
  - ☐ Anti-commutativity of cross product: $\vec{\boldsymbol{x}} \times \vec{\boldsymbol{y}} = -\vec{\boldsymbol{y}} \times \vec{\boldsymbol{x}}$
  - ☐ Bi-linearity of the Cross Product
    - ☐ Linearity in Left Argument: $(\vec{\boldsymbol{x}} + \vec{\boldsymbol{y}}) \times \vec{\boldsymbol{z}} = \vec{\boldsymbol{x}} \times \vec{\boldsymbol{z}} + \vec{\boldsymbol{y}} \times \vec{\boldsymbol{z}}$
    - ☐ Linearity in Right Argument: $\vec{\boldsymbol{x}} \times (\vec{\boldsymbol{y}} + \vec{\boldsymbol{z}}) = \vec{\boldsymbol{x}} \times \vec{\boldsymbol{y}} + \vec{\boldsymbol{x}} \times \vec{\boldsymbol{z}}$
  - ☐ Homogeneity of the Cross Product: $(\alpha\, \vec{\boldsymbol{x}}) \times \vec{\boldsymbol{y}} = \vec{\boldsymbol{x}} \times (\alpha\, \vec{\boldsymbol{y}}) = \alpha\, (\vec{\boldsymbol{x}} \times \vec{\boldsymbol{y}})$
- ☐ State, prove and apply parallelity theorem for cross product: $\vec{\boldsymbol{x}} \parallel \vec{\boldsymbol{y}} \iff \vec{\boldsymbol{x}} \times \vec{\boldsymbol{y}} = 0$
- ☐ State, prove and apply orthogonality theorem for cross product: $\vec{\boldsymbol{x}} \perp (\vec{\boldsymbol{x}} \times \vec{\boldsymbol{y}}) \Leftrightarrow \vec{\boldsymbol{x}} \cdot (\vec{\boldsymbol{x}} \times \vec{\boldsymbol{y}}) = 0$
- ☐ State, prove and apply the Sine theorem for cross product: $\| \vec{\boldsymbol{x}} \times \vec{\boldsymbol{y}} \|_2 = \| \vec{\boldsymbol{x}} \|_2 \, \| \vec{\boldsymbol{y}} \|_2 \, \text{Sin}(\, \theta\,)$
- ☐ Discuss the two important utilities of the cross product
  - ☐ Explain how the cross product is used to produce an output that is orthogonal to the two inputs
  - ☐ Explain how the cross product is used as a measurement of perpendicularity

LESSONS 5: LINES AND CURVES IN $\mathbb{R}^n$

- ☐ If f is a single-variable, real-valued function, explain the notation: $f\ :D \longrightarrow \mathbb{R}$ where $D \subseteq \mathbb{R}$
- ☐ If f is a multivariable, real-valued function, explain the notation: $f\ :D \longrightarrow \mathbb{R}$ where $D \subseteq \mathbb{R}^n$
- ☐ If $\vec{\boldsymbol{r}}$ is a single-variable, vector-valued function, explain the notation: $\vec{\boldsymbol{r}}\ :D \longrightarrow \mathbb{R}^n$ with $D \subseteq \mathbb{R}$
- ☐ State and apply the description of a vector-valued function using parametric equations
- ☐ Transform point-slope form of an equation for a line in $\mathbb{R}^2$ into a vector-valued function $\vec{\boldsymbol{r}}(t)$
  - ☐ Use slope $m = \frac{b}{a}$ to create a direction vector $\vec{\boldsymbol{v}}$ for the line
  - ☐ Derive general vector-valued equation for line in $\mathbb{R}^2$: $\vec{\boldsymbol{r}}(t) = \vec{\boldsymbol{r}}_{\mathbf{0}} + t\ \cdot\ \vec{\boldsymbol{v}}$
  - ☐ Transform vector-valued equation for line into parametric form
  $$\vec{\boldsymbol{r}}(t) = \vec{\boldsymbol{r}}_{\mathbf{0}} + t\ \cdot\ \vec{\boldsymbol{v}} =< x(t)\ , y(t) >$$
- ☐ Demonstrate how the Pythagorean theorem is used to general the implicit equation for a circle with center at point (h, k) and radius r given by
$$\frac{(x-h)^2}{r^2} + \frac{(y-k)^2}{r^2} = 1$$
- ☐ Transform the implicit equation for a circle with center at point (h, k) and radius r into a vector-valued function $\vec{\boldsymbol{r}}(t) =< x(t)\ , y(t) >$
  - ☐ Use a diagram of circle and your algebra/trig skills to demonstrate that
  $$x(t) = h + r * \cos(t) \text{ and } y(t) = k + r * \sin(t)$$
- ☐ Generalize the implicit equation for a circle with center at point (h, k) and radius r to get the implicit equation for an ellipse given by
$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$
  - ☐ Given values of h, k, a, b, graph an ellipse
- ☐ State the vector-valued function $\vec{\boldsymbol{r}}(t) =< x(t)\ , y(t) >$ for an ellipse
- ☐ State and apply the general vector-valued equation for line in $\mathbb{R}^3$: $\vec{\boldsymbol{r}}(t) = \vec{\boldsymbol{r}}_{\mathbf{0}} + t\ \cdot\ \vec{\boldsymbol{v}}$
  - ☐ Transform vector-valued equation for line into parametric form
  $$\vec{\boldsymbol{r}}(t) = \vec{\boldsymbol{r}}_{\mathbf{0}} + t\ \cdot\ \vec{\boldsymbol{v}} =< x(t)\ , y(t), z(t) >$$
  - ☐ Transform vector-valued equation for line in $\mathbb{R}^3$into symmetric equation for a line in $\mathbb{R}^3$
  $$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$
- ☐ Change the domain of vector-valued equation for line in $\mathbb{R}^3$: $\vec{\boldsymbol{r}}(t) = \vec{\boldsymbol{r}}_{\mathbf{0}} + t\ \cdot\ \vec{\boldsymbol{v}}$ to create a line segment connecting two points in $\mathbb{R}^3$

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LESSONS 6: PLANES AND SURFACES IN $\mathbb{R}^3$

☐ Define an explicit (function) equation
- ☐ Identify the general form of an explicit function equation for a curve in $\mathbb{R}^2$: $y = f(x)$
- ☐ Identify the general form of an explicit function equation for a surface in $\mathbb{R}^3$: $z = f(x, y)$

☐ Define an implicit (relation) equation
- ☐ Identify the general form of an implicit (relation) equation for a curve in $\mathbb{R}^2$: $F(x, y) = 0$
- ☐ Identify the general form of an implicit (relation) equation for a surface in $\mathbb{R}^3$: $F(x, y, z) = 0$

☐ Explain the similarities and differences between explicit and implicit equations

☐ Derive an implicit equation for a line in $\mathbb{R}^2$ using a specific point on the line, the dot product and a normal vector to the line
- ☐ Demonstrate how this equation leads to the standard (implicit) equation for a line in the form

$$Ax + By + C = 0$$

☐ Derive an implicit equation for a plane in $\mathbb{R}^3$ using a specific point on the plane, the dot product and a normal vector to the plane
- ☐ Demonstrate how this equation leads to the standard (implicit) equation for a plane in the form

$$ax + by + cz = d$$

☐ Apply equation of plane to solve problems involving planes including (but not limited to):
- ☐ Find the equation of a plane passing through given points in a stated direction
- ☐ Find when planes are parallel or orthogonal
- ☐ Find the distance from a point to a plane
- ☐ Find the parametric equation for a line in the intersection of two planes

☐ State the definition of a quadratic surface in $\mathbb{R}^3$ and the general form for the implicit equation that defines such a surface

☐ State the definition of the graph of a surface in $\mathbb{R}^3$

☐ Discuss three main tools we use to begin graphing surfaces in $\mathbb{R}^3$
- ☐ Use input/output analysis to find valid values of x, y, z
- ☐ Find intercepts with coordinate axis
- ☐ Find traces of surface intersected with specific planes

☐ State and apply the general implicit equation for an ellipsoid in $\mathbb{R}^3$
- ☐ Given values of h, k, m, a, b, c, graph an ellipsoid
- ☐ Find the valid input/outputs, intercepts and traces of an ellipsoid

☐ State and apply the general explicit equation for an elliptic paraboloid in $\mathbb{R}^3$
- ☐ Given values of h, k, a, b, graph an elliptic paraboloid in $\mathbb{R}^3$
- ☐ Find the valid input/outputs, intercepts and traces of an elliptic paraboloid in $\mathbb{R}^3$

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LESSONS 7: GRAPHS AND LEVEL CURVES

- ☐ State the definition of a two-variable function $z = f(x, y)$ where $f: D \subseteq \mathbb{R}^2 \to \mathbb{R}$
- ☐ State the definition of the domain $D \subseteq \mathbb{R}^2$ of a two-variable function $z = f(x, y)$
- ☐ State the definition of the range $\mathrm{Rng}(f) \subseteq \mathbb{R}$ of a two-variable function $z = f(x, y)$
- ☐ Recall the definition of the graph of a surface in $\mathbb{R}^3$ as a set of ordered triplets
- ☐ Apply the definition two-variable function $z = f(x, y)$ where $f: D \subseteq \mathbb{R}^2 \to \mathbb{R}$
    - ☐ Given an explicit equation $z = f(x, y)$, find the domain of the function $D \subseteq \mathbb{R}^2$
    - ☐ Given an explicit equation $z = f(x, y)$, find the range of this function $\mathrm{Rng}(f) \subseteq \mathbb{R}$
- ☐ Let $z = f(x, y)$ and constant $z_0 \in \mathrm{Rng}(f)$. Then, state the definition of a contour curve $C_{z_0}(f)$
- ☐ Apply the definition of contour curve $C_{z_0}(f)$:
    - ☐ Given function $z = f(x, y)$ and constant, find and graph contour curve $C_{z_0}(f) \subseteq \mathbb{R}^3$
    - ☐ Write contour curve contour $C_{z_0}(f)$ as a vector-valued function $\vec{\boldsymbol{r}}(t) =< x(t)\,, y(t), z_0 >$
- ☐ Let $z = f(x, y)$ and constant $z_0 \in \mathrm{Rng}(f)$. Then, state the definition of a level curve $L_{z_0}(f)$
- ☐ Apply the definition of contour curve contour $L_{z_0}(f)$:
    - ☐ Given function $z = f(x, y)$ and constant, find and graph level curve $L_{z_0}(f) \subseteq \mathbb{R}^2$
    - ☐ Write level curve $L_{z_0}(f)$ as a vector-valued function $\vec{\boldsymbol{r}}(t) =< x(t)\,, y(t) >$
    - ☐ Use implicit differentiation to find vector-valued equation for tangent line to $L_{z_0}(f)$

LESSONS 8: LIMITS AND CONTINUITY

- ☐ State the definition of the limit of a function of two variables:

$$\lim_{(x,y) \to (a,b)} f(x, y) = L$$

- ☐ State and apply the theorem for limits of constant and linear two-variable functions
- ☐ State and apply the limit laws for two-variable functions
- ☐ State and apply the two-path test for nonexistence of the limit of a two-variable function:
    - ☐ Focus on the five paths discussed in class including:
        - ☐ Path 1: Along the line $y = b$ in domain with $(x, b) \to (a, b)$
        - ☐ Path 2: Along the line $x = a$ in domain with $(a, y) \to (a, b)$
        - ☐ Path 3: Along the line $y = m(x - a) + b$ in domain with $(x, y) \to (a, b)$
        - ☐ Path 4: Along the curve $y = (x - a)^2 + b$ in domain with $(x, y) \to (a, b)$
        - ☐ Path 5: Along the curve $x = (y - b)^2 + a$ in domain with $(x, y) \to (a, b)$
- ☐ State and apply the guidelines for finding non-obvious limits for two-variable functions
- ☐ State and apply the definition for continuity of a two-variable function at a point (a, b)

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LESSONS 9: PARTIAL DERIVATIVES

☐ Discuss the five-step process for constructing derivatives

☐ Explain each of the five-step used to construct partial derivatives at a point (a, b)

☐Derive the limit definition of the partial derivative in slope notation given by

$$f_x(a,b) = \lim_{x \to a} \frac{f(x,b) - f(a,b)}{x - a}$$

☐ Discuss how the limit definition of $f_x(a, b)$ relates to Path 1 from Lesson 8

☐ Create the limit definitions of the partial derivatives in derivative notation given by

$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

☐Derive the limit definition of the partial derivative in slope notation given by

$$f_y(a,b) = \lim_{y \to b} \frac{f(a,y) - f(a,b)}{y - b}$$

☐ Discuss how the limit definition of $f_y(a, b)$ relates to Path 1 from Lesson 8

☐ Create the limit definitions of the partial derivatives in derivative notation given by

$$f_x(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

☐ State and apply the definition of the partial derivative functions $f_x(x, y)$ and $f_y(x, y)$

☐ Use limit definition of partial derivatives to find $f_x(a, b)$ and $f_y(a, b)$ for given function $z = f(x, y)$

☐ Use the rules for partial derivatives to find $f_x(a, b)$ and $f_y(a, b)$ for given function $z = f(x, y)$

☐ State and apply the definitions of second-order partial derivative functions

☐ $f_{xx}(x,y) = (f_x)_x(x,y) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x} f(x,y)\right] = \frac{\partial^2}{\partial x^2} f(x,y)$

☐ $f_{xy}(x,y) = (f_x)_y(x,y) = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x} f(x,y)\right]$

☐ $f_{yx}(x,y) = (f_y)_x(x,y) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y} f(x,y)\right]$

☐ $f_{yy}(x,y) = (f_y)_y(x,y) = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y} f(x,y)\right] = \frac{\partial^2}{\partial y^2} f(x,y)$

☐ State and apply Clairaut's Theorem that with certain assumptions, we know: $f_{xy}(x, y) = f_{yx}(x, y)$

LESSONS 10: THE CHAIN RULE

- ☐ State the chain rule from Math 1A for ordinary derivatives
- ☐ Properly apply chain rule from Math 1A to differentiate a single-variable composite function
- ☐ State the chain rule for a multivariable function with two intermediate variables and one independent variable.
- ☐ Explain why the chain rule for a multivariable function with two intermediate variables and one independent variable results in an ordinary derivative (not a partial derivative)
- ☐ Given a multivariable composite function $z = f(x(t), y(t))$ with two intermediate variables $x = x(t)$, $y = y(t)$ and one independent variable t, properly apply chain rule to find the ordinary derivative: $\frac{dz}{dt} = \frac{d}{dt}[f(x, y)]$

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LESSON 11: DIRECTIONAL DERIVATIVES AND THE GRADIENT

- ☐ Let $y = f(x)$ is a single-variable function and support $\left(a, f(a)\right)$ is a point on the graph of this function. Using this information, derive the limit definition of the ordinary derivative in slope notation, given by
$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$
- ☐ Provide a detailed the graphical interpretation of the limit definition of the ordinary derivative in slope notation. In particular, identify the two main components of the ordinary derivative:
  - Which part of this definition measures the slope of a secant line?
  - How is the limiting process used to transform a secant line into a tangent line?
  - What does the scalar $f'(a)$ measure?
- ☐ Let $y = f(x)$ is a single-variable function and support $\left(a, f(a)\right)$ is a point on the graph of this function. Using this information, derive the limit definition of the ordinary derivative in derivative notation, given by
$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$
- ☐ Make an explicit connection between the two equivalent definitions of the ordinary derivatives
- ☐ Given a multivariable function $f(x, y)$ and a parameterized line $\vec{\ell}(h) = t \cdot \vec{u} + \vec{p}_0$, derive the limit definition of the directional derivative of f in the direction of unit vector $\vec{u}$.
- ☐ Make sure that you can explicitly explain how this directional derivative uses the same two main components as the definition of the ordinary derivative: namely the slope of a secant line and the limiting process.
- ☐ Derive the dot product formula for the directional derivative, given as
$$D_u f(x, y) = \nabla f(x, y) \cdot \vec{\boldsymbol{u}}$$
by applying the appropriate multivariable chain rule.
- ☐ Given a two- or three-variable function f, find the directional derivative of that function in the direction of a vector $\vec{\boldsymbol{v}}$ (not necessarily a unit vector).
- ☐ State the definition of the gradient of a two- or three-variable function
- ☐ Given a two- or three-variable function f, find the gradient of this function, denoted as $\nabla f(x, y)$.
- ☐ Show that since we assume that $\vec{\boldsymbol{u}}$ is a unit vector and we have $D_u f(x, y) = \nabla f(x, y) \cdot \vec{\boldsymbol{u}}$, we know by the cosine for the dot product that
$$D_u f(x, y) = \|\nabla f(x, y)\| \cdot \cos(\theta)$$
where θ is the angle between the vectors $\nabla f(x, y)$ and $\vec{\boldsymbol{u}}$.
- ☐ Use the fact that $D_u f(x, y) = \|\nabla f(x, y)\| \cdot \cos(\theta)$ to explain which unit direction vectors $\vec{\boldsymbol{u}}$ produce the steepest ascent, steepest decent and no change along the surface.
- ☐ Explain how the steepest ascent, decent, and no change directions are related to the level curves of a surface in $\mathbb{R}^2$.

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LESSON 12: TANGENT PLANES

- ☐ Properly state equation for tangent plane to surface $F(x, y, z) = 0$.
- ☐ Properly state equation for tangent plane to surface $z = f(x, y)$.
- ☐ Given the equation for a surface and a point on that surface, find the equation for a tangent plane on that surface. $z = f(x, y)$.
- ☐ Given the equation for a surface, find all point(s) on the surface whose tangent planes point in the direction of a given vector $\vec{\boldsymbol{v}} \in \mathbb{R}^3$

LESSON 13: MAXIMUM AND MINIMUM VALUES

- ☐ Given a multivariable function, properly state the definitions of a local minimum value and a local minimizer.
- ☐ Given a multivariable function, properly state the definitions of a local maximum value and a local maximum.
- ☐ Properly state the definition of a critical point for a two-variable function.
- ☐ Properly state and apply the second partial derivative test for finding the local maximum and minimum values of a two variable function.
- ☐ Use the second partial derivative test to find the point(s) on a given surface that are closest to some given point in $\mathbb{R}^3$

LESSON 14: LAGRANGE MULTIPLIERS

- ☐ State the definition of a two- or three-variable constrained optimization problem.
- ☐ State the parallel gradients theorem 12.15.
- ☐ State the method of Lagrange Multipliers in Two Variables
- ☐ Apply the method of Lagrange Multipliers in two variables to solve a two-variable constrained optimization problem.
- ☐ Properly state the method of Lagrange Multipliers in three variables
- ☐ Apply the method of Lagrange Multipliers in three variables to solve a two-variable constrained optimization problem.

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Introduction to Numerical Analysis (INA)

INA LESSON 1: SEQUENCES

- ☐ Explain why we are studying sequences and series in Part II of this class:
  - What, specifically, will we use this theory to do?
  - What type of problems will we solve using this theory?
- ☐ State the definition of a sequence
- ☐ State the three different representations of a sequence
- ☐ Given one representation of a sequence, produce the other two representations
- ☐ Use the limit laws for sequences to find the limit of a sequence
- ☐ State the definition of a geometric sequence
- ☐ State the limit of a geometric sequence depending on the value of the ratio r

INA LESSON 2: INFINITE SERIES

- ☐ State the definition of a sequence of partial sums
- ☐ State the definition of an infinite series
- ☐ State the definition of a convergent series
- ☐ State the definition of a divergent series
- ☐ Derive the geometric sum formula
- ☐ Use the geometric sum formula to derive the Geometric Series test
- ☐ Apply Geometric Series test to determine the convergence behavior of a given geometric series.
- ☐ Make sure you know how to recognize a telescoping series
- ☐ Properly apply Telescoping Series technique to determine the convergence behavior of a given telescoping series

INA LESSON 3: DIVERGENCE AND INTEGRAL TESTS

- ☐ State the Divergence test
- ☐ Apply the divergence test to determine the convergence behavior of a given series
- ☐ Prove that the harmonic series, given by $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges
- ☐ State the Integral test
- ☐ Apply the integral test to determine the convergence behavior of a given series
- ☐ Use the integral test to derive the p-series test
- ☐ State the estimating series with positive terms theorem
- ☐ Apply the estimating series with positive terms theorem to estimate the limit of a series
- ☐ State the properties of convergent series
- ☐ Apply the properties of convergent series to find the limit of a series

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INA LESSON 4: RATIO, ROOT AND COMPARISON TESTS

☐ State the ratio test
☐ Apply the ratio test to determine the convergence behavior of a given series
☐ State the root test
☐ Apply the root test to determine the convergence behavior of a given series
☐ State the direct comparison test
☐ Apply the direct comparison test to determine the convergence behavior of a given series
☐ State the limit comparison test
☐ Apply the limit comparison test to determine the convergence behavior of a given series
☐ State the guidelines for choosing a test for series containing positive terms
☐ Use the guidelines for choosing a test for series containing positive terms to determine the convergence behavior of a given series (make sure you understand when to apply each test and which the types of series that test is designed to analyze)

INA LESSONS 5: ALTERNATING SERIES TEST

☐ State the alternating series test.
☐ Apply the alternating series test to determine the convergence behavior of a given series.
☐ State the Remainder theorem in alternating series theorem
☐ Apply the Remainder theorem in alternating series to estimate the limit of a series
☐ State the definition of absolute convergence of an alternating series.
☐ State the definition of conditional convergence of an alternating series.
☐ Give an example of a conditionally convergent series that is not absolutely convergent.
☐ State the Absolute convergence implied conditional convergence theorem
☐ Use the appropriate series test to test if an alternating series is absolutely convergent.
☐ State, in your own words, the conclusions of the Special Series and Convergence Tests.
☐ Given any series, figure out which test to use and then properly apply that test to discuss the convergence behavior of that series.

INA LESSONS 6: APPROXIMATING FUNCTIONS WITH POLYNOMIALS

☐ State the definition of a power series
☐ State the definition of the nth degree Taylor series polynomial $T_n(x)$.
☐ State the definition of the remainder of nth degree Taylor series remainder $R_n(x)$.
☐ Explain the connections between the Taylor series remainder $R_n(x)$, the convergence tests for series, and the implications of what this means for approximating continuous functions using polynomials.

INA LESSONS 7: PROPERTIES OF POWER SEIRES

☐ State the definition of a power series centered at a
☐ State the definition of a the interval of convergence of a power series
☐ State the definition of the radius of convergence of a power series
☐ Given a power series, find the radius of convergence.
☐ State the Convergence of a Power Series Theorem 9.3 p. 678
☐ Use tests for convergence to find the radius and interval of convergence of a given power series
☐ State the Combining power series theorem 9.4 p. 679
☐ Apply the Combining power series theorem 9.4 p. 679
☐ State the Differentiating and Integrating power series theorem 9.5 p. 680
☐ Apply the Differentiating and Integrating power series theorem 9.5 p. 680

INA LESSONS 8: TAYLOR SERIES

☐ State the definition of a Taylor series for a function
☐ State the definition of a MacClaurin series for a function
☐ Prove the Convergence of Taylor Series theorem from first principles
☐ State the binomial coefficient definition s
☐ Prove Binomial series theorem from first principles
☐ Derive, from first principles, the Taylor series for each of the function below and provide the appropriate radius of convergence for each one:
- ☐ $f(x) = \sin(x)$
- ☐ $f(x) = \cos(x)$
- ☐ $f(x) = \tan^{-1}(x)$
- ☐ $f(x) = e^x$
- ☐ $f(x) = \ln(x)$
- ☐ $f(x) = \frac{1}{1+x}$
- ☐ $f(x) = \frac{1}{1-x}$
- ☐ $f(x) = \sqrt[n]{x}$ for $n = 2, 3, 4, 5, 6$

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INA LESSONS 9: WORKING WITH TAYLOR SERIES

☐ Discuss what happens when you the derivative of the Taylor series for each of the following functions:

- ☐ $f(x) = \sin(x)$
- ☐ $f(x) = \cos(x)$

☐ Use Taylor series to approximate the integral $\int_0^1 e^{-x^2}\,dx$

☐ Use Taylor series to approximate the value of a function at a given point.

☐ Apply Taylor series to analyze and model the behavior of real-world phenomenon like:

- ☐ Small-angle motion of a pendulum
- ☐ Bessel Functions to study wave propagation in circular geometries
- ☐ Error functions for the study of the normal distribution in statistics
- ☐ Find your own applications of Taylor series and report back

INA LESSONS 10: INTRODUCTION TO FOURIER SERIES

☐ TBD