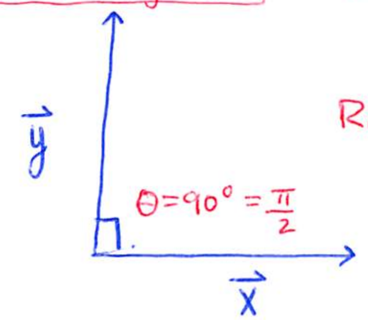
MIC, Lesson 3: In-Class Problems

2. Find values of $b \in \mathbb{R}$ such that the vectors $\langle -11, b, 2\rangle$ & $\langle b, b^2, b\rangle$ are orthogonal.

orthogonal ↓ perpendicular

b ↓ Scalar

Great diagram!! ☺



Let vector $\vec{x} = \langle -11, b, 2\rangle$

$\vec{y} = \langle b, b^2, b\rangle$

Given $\vec{x}$ & $\vec{y} \in \mathbb{R}^3$

Recall the property of a dot/inner product related to orthogonality.

$\vec{x} \cdot \vec{y} = 0$

denotes vectors $\vec{x} \cdot \vec{y}$ orthogonal to one another

Jeff's ideas for improvement

- [ ] I might also write a small note to myself:

Recall: Let θ be the angle between vectors $\vec{x}, \vec{y} \in \mathbb{R}^3$. Remember, we say $\vec{x}$ is orthogonal to $\vec{y}$ iff $\theta = 90°$.

We write this in symbols as $\vec{x} \perp \vec{y}$.

$\vec{x} \cdot \vec{y} = \langle -11, b, 2\rangle \cdot \langle b, b^2, b\rangle$

$= (-11) \cdot b + (b) \cdot b^2 + (2) \cdot b$

$= -11b + b^3 + 2b$

$= b^3 - 9b$

Setting this equal to zero (0)

$b^3 - 9b = 0$

$\underbrace{b}_{A}\underbrace{(b^2 - 9)}_{B} = 0$

So

① $\boxed{b = 0}$

② $b^2 - 9 = 0$

$b^2 = 9$

$b = \pm\sqrt{9}$

$\boxed{b = -3}$

$\boxed{b = 3}$

(2) when $\cos(\theta) = 0$

$$\|x\|_2 \cdot \|y\|_2 \cdot \cos\theta = \|x\|_2 \cdot \|y\|_2 \cdot 0$$
$$= 0$$

$$0 < \|x\|_2 \cdot \|y\|_2$$

(3) when $\cos(\theta)$ is between -1 & 0, so

$-1 \leq \cos(\theta) < 0$

in this case where $\cos(\theta) = -1$

$$\|x\|_2 \cdot \|y\|_2 \cdot \cos(\theta) = -1 \cdot \|x\|_2 \cdot \|y\|_2$$

$$-\|x\|_2 \cdot \|y\|_2 < \|x\|_2 \cdot \|y\|_2 \text{ iff } \cos(\theta) = -1$$

otherwise, when $-1 < \cos(\theta) < 0$

$$\|x\|_2 \cdot \|y\|_2 \cdot \cos(\theta) < \|x\|_2 \cdot \|y\|_2$$

Evaluating all these cases, in all of them

$$\|x\|_2 \cdot \|y\|_2 \cdot \cos(\theta) \leq \|x\|_2 \cdot \|y\|_2$$

Recall

$$\|x\|_2 \cdot \|y\|_2 \cdot \cos(\theta) = |x \cdot y|$$

Therefore,

True

$$\boxed{|x \cdot y| \leq \|x\|_2 \|y\|_2}$$

Jeff's idea's for improvement

- I love that you broke up the inequality into 3 cases! This is a really good idea and makes your work easy to read... One suggestion I have: Clearly label each case and use formatting to call attention to the breaks. For example, you might do as such:

$\boxed{\text{Case 1: } 0 < \cos(\theta) \leq 1}$

Blah, blah, blah...

$\boxed{\text{Case 2: } \cos(\theta) = 0}$

such and such...

$\boxed{\text{Case 3: } -1 \leq \cos(\theta) < 0}$

so on and so forth...

Fun Challenge: Did you know that for a constant $a \in \mathbb{R}$, we have

1. $|a| \leq 1 \iff -1 \leq a \leq 1$
2. If $a \leq b$ and $c \geq 0$, then $ac \leq bc$.

Can you use #1 & #2 to solve this problem differently?

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Let's keep counting pages

MIC, Lesson 3: In-Class Problems

1. If $\underline{x, y \in \mathbb{R}^3}$, then $|x \cdot y| \leq \|x\|_2 \|y\|_2$

x, y are vectors in $\mathbb{R}^3$

Jeff's ideas for improvement

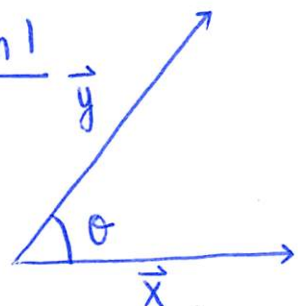
First of all: Wow! This is a really great start. I love how easy it is to read the work and that you obviously took great care in solving these problems. In red, I will provide some ideas for improvement.

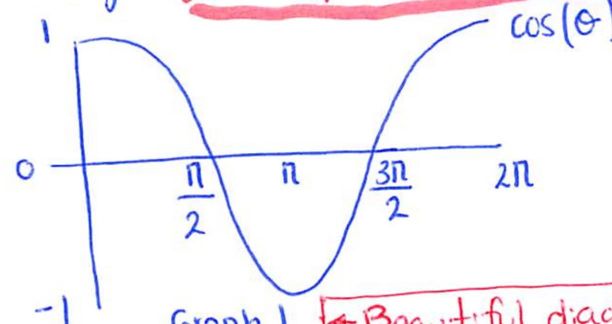
Recall, the cosine formula for the dot product

$$\vec{x} \cdot \vec{y} = \|x\|_2 \|y\|_2 \cdot \cos(\underline{\theta})$$

θ is angle between $\vec{x}$ & $\vec{y}$

Diagram 1





Graph 1 ← Beautiful diagram

Given $\vec{x} \cdot \vec{y} = \|x\|_2 \|y\|_2 \cdot \cos(\theta)$

The question can now be rewritten as

$$\underbrace{\|x\|_2 \cdot \|y\|_2}_{A} \cdot \underbrace{\cos(\theta)}_{B} \underbrace{\leq}_{\text{is less than or equal to}} \underbrace{\|x\|_2 \cdot \|y\|_2}_{A}$$

Let $\|x\|_2 \cdot \|y\|_2 = A$

and

$\cos(\theta) = B$

As can be seen the 2 sides of the equation are the same $\underline{\text{APART}}$ from part $B = \cos(\theta)$

Referring to Graph 1, we can see that $\cos\theta$ oscillates between -1 to 1, so

$$-1 \leq \cos(\theta) \leq 1$$

Given this, let us evaluate 3 possible cases.

① when $\cos\theta$ is between 0 & 1, so

$0 < \cos(\theta) \leq 1$

$\|x\|_2 \cdot \|y\|_2 \cdot \cos(\theta) = \|x\|_2 \cdot \|y\|_2$ iff $\cos(\theta) = 1$

otherwise

$\|x\|_2 \cdot \|y\|_2 \cdot \cos(\theta) < \|x\|_2 \cdot \|y\|_2$ if $0 < \cos(\theta) < 1$

MIC: In- Class Problems

Let's check these solutions
using our dot/inner product
$\vec{x} \cdot \vec{y} = b^3 - 9b$
where
$b^3 - 9b = 0$

1. when $b = 0$

$(0)^3 - 9(0) = 0$

$0 - 0 = 0$

$0 = 0$

2. When $b = 3$

$(3)^3 - 9(3) = 27 - 27$

$= 0$

3. when $b = -3$

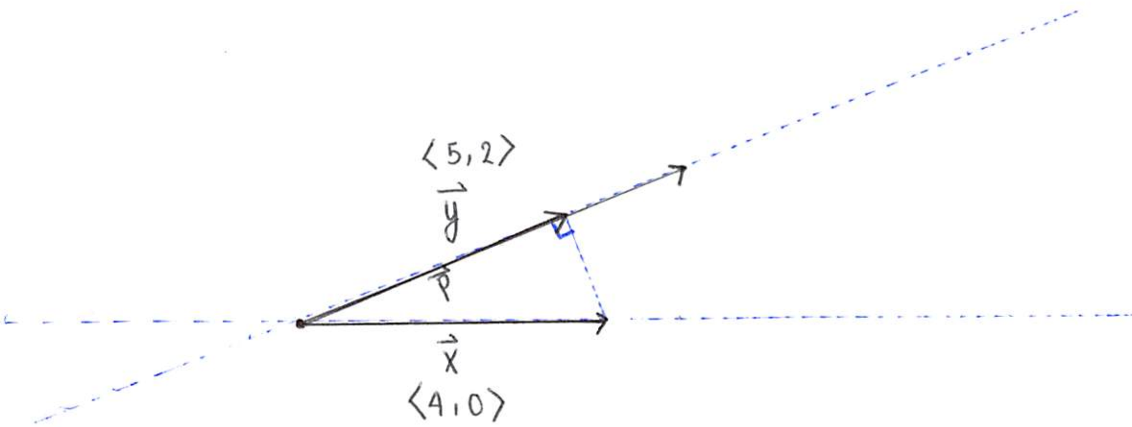
$(-3)^3 - 9(-3) = -27 + 27$

$= 0$

So the solutions for possible values of $b = 0, 3, -3$

Math 1C, Lesson 3: In-Class Problems

3. Given $x = \langle 4,0 \rangle$ & $y = \langle 5,2 \rangle$, find the projection of vector x onto vector y



We will try to find $\vec{p}$

$\downarrow$

projection of $\vec{x}$ onto $\vec{y}$

Recall lesson 3.11: Projections via Dot Product

$\vec{p} = \alpha \circ \vec{y}$

where α is an unknown constant and $\alpha \in \mathbb{R}$

To find α

$$\alpha = \frac{\vec{x} \circ \vec{y}}{\|\vec{y}\|_2}$$

Let us find $\|\vec{y}\|_2$

$$\|\vec{y}\|_2^2 = 5^2 + 2^2$$
$$= 25 + 4$$
$$= 29$$

$$\|\vec{y}\|_2 = \sqrt{29}$$

Let us find $\vec{x} \cdot \vec{y}$

$$\vec{x} \cdot \vec{y} = \langle 4,0 \rangle \circ \langle 5,2 \rangle$$
$$= 4 \circ 5 + 0 \circ 2$$
$$= 20 + 0$$
$$= 20$$

So

$$\alpha = \frac{20}{\sqrt{29}}$$

So $\vec{p} = \alpha \circ \vec{y}$

$$= \frac{20}{\sqrt{29}} \circ \langle 5, 2 \rangle$$

$$= \left\langle \frac{100}{\sqrt{29}}, \frac{40}{\sqrt{29}} \right\rangle$$

MIC, Lesson 3: In-Class Problems

4. $V = -4i - j + 4k$

$w = i + 3j + 2k$

so $\vec{v} = \langle -4, -1, 4 \rangle$

$\vec{w} = \langle 1, 3, 2 \rangle$

and $\vec{v}$ & $\vec{w} \in \mathbb{R}^3$

Find $\vec{v} \cdot \vec{w}$

$$\vec{v} \cdot \vec{w} = \langle -4, -1, 4 \rangle \cdot \langle 1, 3, 2 \rangle$$
$$= (-4) \cdot 1 + (-1) \cdot 3) + (4) \cdot 2$$
$$= -4 - 3 + 8$$
$$= \boxed{1}.$$

To find the angle between $\vec{v}$ & $\vec{w}$

Recall,

$$\vec{x} \cdot \vec{y} = \|\vec{x}\|_2 \cdot \|\vec{y}\|_2 \cdot \cos(\theta)$$

where θ = angle between $\vec{v}$ & $\vec{w}$

Let $\vec{x} = \vec{v}$ and $\vec{y} = \vec{w}$

Let us find $\|\vec{x}\|_2$

$$\|\vec{x}\|_2^2 = (-4)^2 + (-1)^2 + (4)^2$$
$$= 16 + 1 + 16$$
$$= 33$$
$$\|\vec{x}\|_2 = \sqrt{33}$$

Let us find $\|\vec{w}\|_2$

$$\|\vec{w}\|_2^2 = 1^2 + 3^2 + 2^2$$
$$= 1 + 9 + 4$$
$$= 14$$
$$\|\vec{w}\|_2 = \sqrt{14}$$

Returning to

$$\vec{v} \cdot \vec{w} = \|\vec{v}\|_2 \cdot \|\vec{w}\|_2 \cdot \cos(\theta)$$

Let's now make the appropriate substitution

$$1 = \sqrt{33} \cdot \sqrt{14} \cdot \cos(\theta)$$

$$\frac{1}{\sqrt{33} \cdot \sqrt{14}} = \cos(\theta)$$

$$\cos^{-1}\left(\frac{1}{\sqrt{33} \cdot \sqrt{14}}\right) = \theta$$

$$\theta \approx 1.52$$