Reference: Brigg's "Calculus: Early Transcendentals, Second Edition"
Topics: Section 8.1 p. 596 - 606 Section 8.2 p. 607 - 619

Definition. *p. 597* ***Sequence***

A **sequence** $\{a_n\}_{n=1}^{\infty}$ is a function $a : \mathbb{N} \longrightarrow \mathbb{R}$. In other words, a sequences is an ordered list of numbers. We can write any sequence as

i. An **ordered set** as $\{a_1, a_2, a_3, a_4, ..., a_n, ...\}$. The term a_1 is called the *first term*, a_2 is called the *second term*, and in general a_n is called the *nth term*.

ii. A **recurrence relation** of the form $a_{n+1} = f(a_n)$ for $n \in \mathbb{N}$ where a_1 must be given.

iii. An **explicit formula** of the form $a_n = f(n)$ for $n \in \mathbb{N}$.

Definition. *p. 599* ***Limit of a Sequence***

If the terms of sequence $\{a_n\}_{n=1}^{\infty}$ approach a unique number L as n increases, we say

$$\lim_{n \to \infty} a_n = L.$$

That is to say, if a_n can be made arbitrarily close to L by taking n "sufficiently" large, then we say the limit of the sequence $\{a_n\}_{n=1}^{\infty}$ is L.

If $\lim_{n \to \infty} a_n = L$, then we say that $\{a_n\}_{n=1}^{\infty}$ **converges** to L.

If the terms of the sequence $\{a_n\}_{n=1}^{\infty}$ do not approach a single number as n increases, we say the sequences has no limit and the sequence **diverges**.

Theorem 8.1. *p. 607* ***Limits of Sequences from Limits of Functions***

Suppose $f(x)$ is a function such that $f(n) = a_n$ for all $n \in \mathbb{N}$. If $\lim_{x \to \infty} f(x) = L$, then the limit of the sequence $\{a_n\}_{n=1}^{\infty}$ is given by $\lim_{n \to \infty} a_n = L$.

Theorem 8.2. *p. 607* ***Limit Laws for Sequences***

Suppose that $c \in \mathbb{R}$ and p is a positive integer. Suppose that the limits

$$\lim_{n\to\infty} a_n = A \qquad \text{and} \qquad \lim_{n\to\infty} b_n = B$$

exist. Then, as long as we check these conditions, we can conclude

1. *Sum Law:* $\lim_{n\to\infty} \big(a_n + b_n\big) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = A + B$

2. *Difference Law:* $\lim_{n\to\infty} \big(a_n - b_n\big) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n = A - B$

3. *Constant Multiple Law:* $\lim_{n\to\infty} (c \cdot a_n) = c \cdot \lim_{n\to\infty} a_n = c \cdot A$

4. *Product Law:* $\lim_{n\to\infty} (a_n \cdot b_n) = \big(\lim_{n\to\infty} a_n\big) \cdot \big(\lim_{n\to\infty} b_n\big) = A \cdot B$

5. *Quotient Law:* $\lim_{n\to\infty} \left[\frac{a_n}{b_n}\right] = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{A}{B}$ if $\lim_{n\to\infty} b_n = B \neq 0$

6. *Constant Law:* $\lim_{n\to\infty} c = c$

7. *General Root Law:* $\lim_{n\to\infty} \left[\ a_n{}^p\ \right] = \left[\ \lim_{n\to\infty} a_n\ \right]^p = A^p$ (if $p > 0$ and $a_n > 0$.)

Definition. *p. 608* ***Terminology for Sequences***

$\{a_n\}_{n=1}^{\infty}$ is **increasing** if $a_{n+1} > a_n$ for all $n \in \mathbb{N}$

$\{a_n\}_{n=1}^{\infty}$ is **nondecreasing** if $a_{n+1} \geq a_n$ for all $n \in \mathbb{N}$

$\{a_n\}_{n=1}^{\infty}$ is **decreasing** if $a_{n+1} < a_n$ for all $n \in \mathbb{N}$

$\{a_n\}_{n=1}^{\infty}$ is **nonincreasing** if $a_{n+1} \leq a_n$ for all $n \in \mathbb{N}$

$\{a_n\}_{n=1}^{\infty}$ is **monotonic** if it is either nonincreasing or nondecreasing

$\{a_n\}_{n=1}^{\infty}$ is **bounded** if there is a real number $M \in \mathbb{R}$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$

Definition. *p. 609* ***Geometric Sequence***

We say that a sequence $\{a_n\}_{n=1}^{\infty}$ is a **geometric sequence** if it has the property that each term a_{n+1} is obtained by multiplying the previous term a_n by a fixed constant r. In other words, geometric sequences have the property that $a_{n+1} = ra_n$. The term r is call the **ratio** of the sequence. Further, all geometric sequences can be written with general formula $a_n = a \cdot r^{n-1}$ for some constant $a \in \mathbb{R}$.

Theorem 8.3. *p. 611* ***Limit of a geometric sequence***

Let $r \in \mathbb{R}$ be a real number. Then

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \\ \text{does not exist} & \text{if } r \leq -1 \text{ or } r > 1 \end{cases}$$

If $r > 0$, then the sequences $\{r^n\}_{n=1}^{\infty}$ is a monotonic sequence. if $r < 0$, then the sequence $\{r^n\}_{n=1}^{\infty}$ oscillates.

Theorem 8.4. *p. 611* ***The Squeeze Theorem***

Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, and $\{c_n\}_{n=1}^{\infty}$ be sequences with $a_n \leq b_n \leq c_n$ for all integers n greater than some index $N \in \mathbb{N}$. If

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L,$$

then $\lim_{n\to\infty} b_n = L$.

Theorem 8.5. *p. 612* ***Bounded Monotonic Sequences***

Every bounded, monotonic sequence is convergent.

Theorem. ***Absolute Value to Conclude Convergence***

If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Theorem. ***Direct Substitution Property***

If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then

$$\lim_{n\to\infty} f(a_n) = f(L).$$

Definition. *p. 615* ***Precise Definition of a Limit of a Sequence***

The sequence $\{a_n\}_{n=1}^{\infty}$ converges to limit L provided that the terms of a_n can be made "arbitrarily" close to L by taking n "sufficiently large." More precisely, sequence $\{a_n\}_{n=1}^{\infty}$ has a unique limit L if and only if, for any $\epsilon > 0$, there is a positive integer $N \in \mathbb{N}$ (that depends only on ϵ) such that if $n > N$, then

$$|a_n - L| < \epsilon$$

If the **limit of a sequence** is L, we say the sequence **converges** to L and we write

$$\lim_{n\to\infty} a_n = L$$

A sequence that does not converge is said to **diverge**.