Lesson 16: Geometric and Telescoping Series Handout
Reference: Brigg's "Calculus: Early Transcendentals, Second Edition"
Section 8.3: Infinite Series, p. 619 - 626

Definition. *p. 602* ***Infinite Series***

Let the sequence $\{a_n\}_{n=1}^{\infty}$ be given. The **sequence of partial sums** $\{S_n\}_{n=1}^{\infty}$ associated with this sequence has the terms

$$\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ S_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ S_n &= a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k \text{ for } n \in \mathbb{N} \end{aligned}$$

The **infinite series** associated with this sequence is the limit of the sequence of partial sums

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$$

Definition. *p.602* ***Limit of an Infinite Series***

If the sequence of partial sums $\{S_n\}_{n=1}^{\infty}$ approaches a limit L, we say

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} S_n = L.$$

If $\lim_{n\to\infty} S_n = L$, then we say the infinite series **converges** to L.

If the sequence of partial sums diverges, we say the infinite series **diverges**.

Definition. *p. 620* ***Geometric Sum Formula***

A **geometric sum** with n terms has the form

$$S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} = \sum_{k=1}^{n} ar^{k-1}$$

Using this formulation, we see that

$$S_n - rS_n = a - ar^n.$$

If $r \neq 1$, we can solve for S_n and produce the general formula for the geometric sum

$$S_n = \boxed{\sum_{k=1}^{n} ar^{k-1} = a\frac{1-r^n}{1-r}}$$

Note: We can write the series using different upper and lower index as follows:

$$\sum_{k=1}^{n} ar^{k-1} = \sum_{k=0}^{n-1} ar^k.$$

Theorem 8.7. *p. 621* ***Geometric Series Test***

Let $a \neq 0$ and let r be a real number. Then, the series

$$\sum_{k=1}^{\infty} ar^{k-1}$$

has the following convergence behavior:

If $|r| < 1$, then the series converges and $\displaystyle\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}$.

If $|r| \geq 1$, then the series diverges.

Note: We can write the series using different upper and lower index as follows:

$$\sum_{k=1}^{\infty} ar^{k-1} = \sum_{k=0}^{\infty} ar^k$$

In both cases, we can determine the convergence behavior based on the geometric sum formula combined with our knowledge of the limits of geometric sequences.

Definition. ***Telescoping Series Technique***

A **telescoping series** is an infinite series whose partial sums eventually only have a fixed number of terms after cancellation. To apply this technique, we need to write each term of the series as a difference of terms from a a specially designed sequence. In particular, if we want to apply the telescoping series technique to series

$$\sum_{n=1}^{\infty} b_n$$

then, we need to write $b_n = a_n - a_{n+1}$ for a properly constructed sequence $\{a_n\}_{n=1}^{\infty}$. If we can do this, then we can use the method of differences will generally lead to analysis that follows the pattern below:

$$\begin{aligned}
\sum_{n=1}^{\infty} b_n &= \lim_{N\to\infty} \sum_{n=1}^{N} \left[\ a_n - a_{n+1}\ \right] \\
&= \lim_{N\to\infty} \left(\left[\ a_1 - a_2\ \right] + \left[\ a_2 - a_3\ \right] + \left[\ a_3 - a_4\ \right] + \cdots + \left[\ a_{N-1} - a_N\ \right] \right) \\
&= \lim_{N\to\infty} \left(a_1 + \left[\ -a_2 + a_2\ \right] + \left[\ -a_3 + a_3\ \right] \cdots + \left[\ -a_{N-1} + a_{N-1}\ \right] - a_N \right) \\
&= \lim_{N\to\infty}\ a_1 - a_N \\
&= a_1 - \lim_{N\to\infty}\ a_N
\end{aligned}$$

Notice that, in the method of differences, the second part of each difference cancels the first part of the next term. The original series converges if and only the limit $\lim_{N\to\infty} a_N$ exists.