Lesson 17: Divergence and Integral Tests Handout
Reference: Brigg's "Calculus: Early Transcendentals, Second Edition"
Topics: Section 8.4: The Divergence and Integral Tests, p. 627 - 640

Theorem 8.8. *p. 627* ***Divergence Test***

If the infinite series $\sum a_k$ converges, then $\lim_{k\to\infty} a_k = 0$.

Equivalently, if $\lim_{k\to\infty} a_k \neq 0$, then the infinite series $\sum a_k$ diverges.

Note: The divergence test cannot be used to conclude that a series converges. This test ONLY provides a quick mechanism to check for divergence.

Definition. *p. 628* ***Harmonic Series***

The famous **harmonic series** is given by $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$

Theorem 8.9. *p. 630* ***Harmonic Series***

The **harmonic series** $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, even though the terms of the series approach zero.

Theorem 8.10. *p. 630* ***Integral Test***

Suppose the function $f(x)$ satisfies the following three conditions for $x \geq 1$:

i. $f(x)$ is continuous

ii. $f(x)$ is positive

iii. $f(x)$ is decreasing

Suppose also that $a_k = f(k)$ for all $k \in \mathbb{N}$. Then

$$\sum_{k=1}^{\infty} a_k \text{ and } \int_1^{\infty} f(x)dx$$

either both converge or both diverge. In the case of convergence, the value of the integral is NOT equal to the value of the series.

Note: The Integral Test is used to determine *whether* a series converges or diverges. For this reason, adding or subtracting a finite number of terms in the series or changing the lower limit of the integration to another finite point does not change the outcome of the test. For this reason, the Integral Test does NOT depend on the lower index of the series or the lower limit of the integral.

Theorem 8.11. *p. 632* ***Convergence of the p-Series (The p-Series Test)***

The **p-series** $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges for all $p > 1$ and diverges for all $p \leq 1$.

Theorem 8.12. *p. 635* ***Estimating Series with Positive Terms***

Let $f(x)$ be a continuous, positive decreasing function, for $x \geq 1$, and define sequence $a_k = f(k)$ for all $k \in \mathbb{N}$. Suppose that the limit of the associated convergent infinite series is

$$S = \sum_{k=1}^{\infty} a_k$$

Define the sequence of partial sums

$$S_n = \sum_{k=1}^{n} a_k$$

as the sum of the first n terms of the series. Then, the remainder $R_n = S - S_n$ satisfies the following inequality:

$$R_n < \int_{n}^{\infty} f(x)dx$$

Furthermore, the exact value of the series satisfies the following bounds:

$$S_n + \int_{n+1}^{\infty} f(x)dx \;<\; \sum_{k=1}^{\infty} a_k \;<\; S_n + \int_{n}^{\infty} f(x)dx$$

Definition. *p. 636* ***Leading Terms and Tail of a Series***

Let N be a positive integer. Given and infinite series

$$\sum_{k=1}^{\infty} a_k$$

the **leading terms** of this series are the terms at the beginning with small index, say all terms a_k with $k < N$. The **tail** of an infinite series consists of the sum of all terms at the "end" of the series with large and increasing index, given by

$$\sum_{k=N}^{\infty} a_k$$

The convergence or divergence behavior of an infinite series depend only on the tail of the series. The value of a convergent series is determined primarily by the leading terms.

Theorem 8.13. *p. 636* ***Properties of Convergent Series***

Suppose that $c \in \mathbb{R}$ and suppose that

$$\sum_{k=1}^{\infty} a_k = A \qquad \text{and} \qquad \sum_{k=1}^{\infty} b_k = B.$$

Then, as long as we check these conditions, we can conclude

1. *Constant Multiple Law:* $\sum_{k=1}^{\infty} c \cdot a_k = c \cdot \sum_{k=1}^{\infty} a_k = c \cdot A$
2. *Sum Law:* $\sum_{k=1}^{\infty} \big(a_n \pm b_n\big) = \sum_{k=1}^{\infty} a_n \pm \sum_{k=1}^{\infty} b_n = A \pm B$
3. *Sum Law:* If M is a positive integer, then

$$\sum_{k=1}^{\infty} a_k \text{ and } \sum_{k=M}^{\infty} a_k$$

either both converge or both diverge. In general, whether a series converges does not depend on a finite number of terms added to or removed from the series. However, the *value* of a convergent series does change if nonzero terms are added or removed.