Lesson 18: Ratio, Root and Comparison Tests Handout
Reference: Brigg's "Calculus: Early Transcendentals, Second Edition"
Topics: Section 8.5: The Ratio, Root, and Comparison Tests, p. 641 - 649

Theorem 8.14. *p. 641* ***Ratio Test***

Let $\sum_{k=1}^{\infty} a_k$ be an infinite series with positive terms $a_k > 0$ for all $k \in \mathbb{N}$. Let

$$r = \lim_{k\to\infty} \frac{a_{k+1}}{a_k}$$

1. If $0 \le r < 1$, then the series converges.

2. If $r > 1$ (including $r = \infty$), then the series diverges.

3. If $r = 1$, then the ratio test is inconclusive.

Note: In words, the ratio test says that the limit of the ratio of successive terms of a positive series must be less than 1 to guarantee convergence of the series.

Theorem 8.15. *p. 642* ***Root Test***

Let $\sum_{k=1}^{\infty} a_k$ be an infinite series with nonnegative terms $a_k \ge 0$ for all $k \in \mathbb{N}$. Let

$$\rho = \lim_{k\to\infty} \sqrt[k]{a_k}$$

1. If $0 \le \rho < 1$, then the series converges.

2. If $\rho > 1$ (including $\rho = \infty$), then the series diverges.

3. If $\rho = 1$, then the root test is inconclusive.

Theorem 8.16. *p. 643* ***The (Direct) Comparison Test***

Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be infinite series with positive terms.

1. If $0 < a_k \le b_k$ for all $k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} b_k$ converge, then the series $\sum_{k=1}^{\infty} a_k$ converges.
2. If $0 < b_k \le a_k$ for all $k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} b_k$ diverge, then the series $\sum_{k=1}^{\infty} a_k$ diverges.

Note: Whether a series converges depends on the behavior of the terms in the tail of the series. Thus, the inequalities

$$0 < a_k \le b_k \text{ and } 0 < b_k \le a_k$$

in this test need not hold for all terms of the series. They must hold for all $k \ge M$ for some positive integer $M \in \mathbb{N}$.

Theorem 8.17. *p. 643* ***The Limit Comparison Test***

Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be infinite series with positive terms. Let

$$\lim_{k\to\infty} \frac{a_k}{b_k} = L$$

1. If $0 < L < \infty$ (that is, L is a positive, finite number), then series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ either both converges or both diverge.
2. If $L = 0$ and $\sum_{k=1}^{\infty} b_k$ converge, then the series $\sum_{k=1}^{\infty} a_k$ converges.
3. If $L = \infty$ and $\sum_{k=1}^{\infty} b_k$ diverges, then the series $\sum_{k=1}^{\infty} a_k$ diverges.

Guidelines for Choosing a Test for Series Containing Positive Terms

Here are some reasonable suggestions when testing a series of positive terms for convergence:

1. Begin with the Divergence Test.
2. Ask yourself: "Is the series a special series?" and make sure you can recall the convergence properties of each of the following special series.
   i. **Geometric series**
   ii. $p-$**series**
   iii. **Telescoping series**
   iv. **Harmonic series**
3. If the general kth term of the series look like a function that you can integrate, then try the integral test. Make sure you remember and can apply techniques of integration including:
   - u-substitution
   - Integration by parts
4. If the general kth term of the series involves $k!, k^k$, or a^k for some $a \in \mathbb{R}$, then try the ratio test. Series with k in the exponent may yield to the Root Test.
5. If the general kth term of the series is a rational function of k (or a root of a rational function), use the Direct Comparison Test or the Limit Comparison test with the families of series given in Step 2 above as comparison series.