Lesson 19: Alternating Series Handout
Reference: Brigg's "Calculus: Early Transcendentals, Second Edition"
Topics: Section 8.6: Alternating Series, p. 649 - 660

Definition. *p. 649* ***Alternating Series***

An **alternating series** is a series in the form

$$\sum_{k=1}^{\infty}(-1)^k a_k \qquad \text{or} \qquad \sum_{k=1}^{\infty}(-1)^{k+1} a_k.$$

where $a_k > 0$ for all $k \in \mathbb{N}$. In this case, the signs of each sequence term strictly alternate from positive to negative. The factor $(-1)^k$ or $(-1)^{k+1}$ has the pattern $\{..., 1, -1, 1, -1, 1, -1, ...\}$ and provides the alternating signs on the sequence of terms.

Theorem 8.18. *p. 650* ***Alternating Series Test***

Let $a_k > 0$ for all $k \in \mathbb{N}$ and consider the alternative series

$$\sum_{k=1}^{\infty}(-1)^{k+1} a_k$$

If we confirm BOTH of the following:

1. The terms of the series are nonincreasing in magnitude ($0 < a_{k+1} \leq a_k$ for k greater than some positive integer M)
2. $\lim_{k\to\infty} a_k = 0$

then the alternating series converges.

Theorem 8.19. *p. 651* ***Alternating Harmonic Series***

The **alternating harmonic series**

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}$$

converges (even though the harmonic series diverges).

Definition. *p. 649* ***Absolute and Conditional Convergence***

If $\sum_{k=1}^{\infty} |a_k|$ converges, then we say $\sum_{k=1}^{\infty} a_k$ **converges absolutely**.

If $\sum_{k=1}^{\infty} |a_k|$ diverges and $\sum_{k=1}^{\infty} a_k$ converges, then we say $\sum_{k=1}^{\infty} a_k$ **converges conditionally**.

Theorem 8.20. *p. 652* ***Remainder in Alternating Series***

Let $S = \sum_{k=1}^{\infty} (-1)^{k+1} a_k$ be a convergent alternating series with terms that are nonincreasing in magnitude. Let $R_n = S - S_n$ be the remainder in approximating the value of the series by the sum of its first n terms. Then

$$|R_n| \ \le \ a_{n+1}$$

In other words, the magnitude of the remainder of a convergent alternating series. is less than or equal to the magnitude of the first neglected term.

Theorem 8.21. *p. 651* ***Absolute Convergence Implies Convergence***

If $\sum_{k=1}^{\infty} |a_k|$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.

Equivalently, if $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} |a_k|$ diverges.

Note: In words, we say that absolute converges implies convergence.

Table 8.4 Special Series and Convergence Tests

Series or test	Form of series	Condition for convergence	Condition for divergence	Comments
Geometric series	$\sum_{k=0}^{\infty} a r^k$, $a \neq 0$	$\lvert r \rvert < 1$	$\lvert r \rvert \geq 1$	If $\lvert r \rvert < 1$, then $\sum_{k=0}^{\infty} a r^k = \frac{a}{1-r}$.
Divergence Test	$\sum_{k=1}^{\infty} a_k$	Does not apply	$\lim_{k\to\infty} a_k \neq 0$	Cannot be used to prove convergence
Integral Test	$\sum_{k=1}^{\infty} a_k$, where $a_k = f(k)$ and f is continuous, positive, and decreasing	$\int_1^{\infty} f(x)\,dx$ converges.	$\int_1^{\infty} f(x)\,dx$ diverges.	The value of the integral is not the value of the series.
p-series	$\sum_{k=1}^{\infty} \frac{1}{k^p}$	$p > 1$	$p \leq 1$	Useful for comparison tests
Ratio Test	$\sum_{k=1}^{\infty} a_k$, where $a_k > 0$	$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} < 1$	$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} > 1$	Inconclusive if $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = 1$
Root Test	$\sum_{k=1}^{\infty} a_k$, where $a_k \geq 0$	$\lim_{k\to\infty} \sqrt[k]{a_k} < 1$	$\lim_{k\to\infty} \sqrt[k]{a_k} > 1$	Inconclusive if $\lim_{k\to\infty} \sqrt[k]{a_k} = 1$
Comparison Test	$\sum_{k=1}^{\infty} a_k$, where $a_k > 0$	$0 < a_k \leq b_k$ and $\sum_{k=1}^{\infty} b_k$ converges	$0 < b_k \leq a_k$ and $\sum_{k=1}^{\infty} b_k$ diverges	$\sum_{k=1}^{\infty} a_k$ is given; you supply $\sum_{k=1}^{\infty} b_k$.
Limit Comparison Test	$\sum_{k=1}^{\infty} a_k$, where $a_k > 0$, $b_k > 0$	$0 \leq \lim_{k\to\infty} \frac{a_k}{b_k} < \infty$ and $\sum_{k=1}^{\infty} b_k$ converges.	$\lim_{k\to\infty} \frac{a_k}{b_k} > 0$ and $\sum_{k=1}^{\infty} b_k$ diverges.	$\sum_{k=1}^{\infty} a_k$ is given; you supply $\sum_{k=1}^{\infty} b_k$.
Alternating Series Test	$\sum_{k=1}^{\infty} (-1)^k a_k$, where $a_k > 0$, $0 < a_{k+1} \leq a_k$	$\lim_{k\to\infty} a_k = 0$	$\lim_{k\to\infty} a_k \neq 0$	Remainder R_n satisfies $\lvert R_n \rvert \leq a_{n+1}$
Absolute Convergence	$\sum_{k=1}^{\infty} a_k$, a_k arbitrary	$\sum_{k=1}^{\infty} \lvert a_k \rvert$		Applies to arbitrary series