Lesson 20: Approximating Functions with Polynomials
Reference: Brigg's "Calculus: Early Transcendentals, Second Edition"
Section 9.1: Approximating Functions with Polynomials, p. 661 - 675

Definition. *p. 661* ***Power Series (Centered at*** a***)***

A **power series** centered at a is an infinite series in the form

$$\sum_{k=0}^{\infty} c_k \, (x-a)^k,$$

where the scalar $a \in \mathbb{R}$ is a constant real number, the sequence terms c_k are constant and x is variable. The sequence terms $\{c_k\}_{k=0}^{\infty}$ are known as the **coefficients** of the power series and scalar a is called the **center** of the power series. This type of series is called a *power* series because it is constructed using powers of $(x-a)$.

Definition. *p. 664* ***Taylor Polynomials***

Let $f(x)$ be a single-variable function with $f', f'', \ldots$ and $f^{(n)}$ defined a input $x = a$. The n**th-order Taylor polynomial** of f with its **center** at a, denoted as $p_n(x)$, has the property that it matches in value, slope and all derivatives up to the nth derivative at a with

$$p_n(a) = f(a), \qquad p_n'(a) = f'(a), \qquad \cdots \qquad p_n^{(n)}(a) = f^{(n)}(a).$$

The nth-order Taylor polynomial centered at a is

$$p_n(x) = f(a) + f'(a)\,(x-a) + \frac{f''(a)}{2!}\,(x-a)^2 + \frac{f^{(3)}(a)}{3!}\,(x-a)^3 + \cdots \frac{f^{(n)}(a)}{n!}\,(x-a)^n$$

$$= \sum_{k=0}^{n} c_k \, (x-a)^k$$

The **coefficients** of this polynomial are given by

$$c_k = \frac{f^{(k)}(a)}{k!}, \qquad \text{for } k \;=\; 0, 1, 2, 3, \ldots, n.$$

Definition. *p. 664* ***Remainder in a Taylor Polynomial***

Let $p_n(x)$ be the Taylor polynomial of order n for the function $f(x)$. The **remainder** in using $p_n(x)$ to approximate f at the point x is

$$R_n(x) = f(x) - p_n(x).$$

Theorem 9.1. *p. 668* ***Taylor's Theorem (Remainder Theorem)***

Let $f(x)$ be a function with continuous derivatives up to $f^{(n+1)}(x)$ on an open interval $I \subseteq \mathbb{R}$ containing the point a. Then, for all $x \in I$ we have

$$f(x) = p_n(x) + R_n(x)$$

where $p_n(x)$ is the nth-order Taylor polynomial for f centered at a and the remainder is

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}\,(x-a)^{n+1},$$

for some point c between x and a.

Theorem 9.2. *p. 669* ***Estimate of the Remainder***

Let n be a fixed positive integer. Suppose there exists a number M such that

$$|\, f^{(n+1)(c)\,|\leq M}$$

for all c between a and x inclusive. The remainder in the nth-order Taylor polynomial for f centered at a satisfies the following inequality

$$|\, R_n(x)\,| = |\, f(x) - p_n(x)\,| \;\leq\; M\,\frac{|\, x-a\,|^{n+1}}{(n+1)!}.$$