Lesson 21: Properties of Power Series Handout
Reference: Brigg's "Calculus: Early Transcendentals, Second Edition"
Section 9.2: Properties of Power Series, p. 675 - 684

Definition. *p. 676* ***Power Series (Centered at a)***

A **power series** has the general form

$$\sum_{k=0}^{\infty} c_k (x-a)^k,$$

where the scalar $a \in \mathbb{R}$ is a constant real number, the sequence terms c_k are constant and x is variable. The sequence terms $\{c_k\}_{k=0}^{\infty}$ are known as the **coefficients** of the power series and scalar a is called the **center** of the power series. The set of all values of variable x for which the series converges is called the **interval of convergence**, denoted as an interval $I \subseteq \mathbb{R}$. The distance from the center of the interval of convergence to the boundary of the interval is called the **radius of convergence** and is denoted by R.

Theorem 9.3. *p. 678* ***Convergence of Power Series***

A power series $\sum_{k=0}^{\infty} c_k (x-a)^k$, centered at a converges in one of three ways:

1. **Infinite Radius of Convergence**

   The series converges for all values of variable $x \in \mathbb{R}$. In this case, the interval of converges is the entire real number line $I = \mathbb{R}$, denoted as interval $(-\infty, \infty)$ and the radius of convergence is $R = \infty$.

2. **Finite, Positive Radius of Convergence**

   There is a real number $R > 0$ such that the series converges for all $|x-a| < R$ and diverges for all $|x-a| > R$. In this case, the radius of convergence is the positive number R.

3. **Zero Radius of Convergence**

   The series converges only at $x = a$ and the radius of convergence is $R = 0$.

Theorem 9.4. *p. 679* ***Combining Power Series***

Suppose the functions $f(x)$ and $g(x)$ can be represented by convergent power series

$$f(x) = \sum_{k=0}^{\infty} c_k\, x^k \qquad \text{and} \qquad g(x) = \sum_{k=0}^{\infty} d_k\, x^k$$

on the interval I.

1. **Sum and Difference Rule**

   The power series $\sum_{k=0}^{\infty} (c_k \pm d_k)\, x^k$ converges to $f(x) \pm g(x)$ on I.

2. **Multiplication by a power function**

   Suppose that $m \in \mathbb{Z}$ with $k + m \geq 0$ for all terms of the power series

   $$x^m \sum_{k=0}^{\infty} c_k\, x^k = \sum_{k=0}^{\infty} c_k\, x^{m+k}$$

   This series converges to $x^m\, f(x)$ for all $x \neq 0$ in I. If $x = 0$, the series converges to

   $$\lim_{x \to 0} x^m\, f(x).$$

3. **Composition**

   If $h(x) = b\, x^m$ for some positive integer $m \in \mathbb{N}$ and a nonzero real number b, then the power series

   $$\sum_{k=0}^{\infty} c_k \left(h(x) \right)^k$$

   converges to the composite function $f(h(x))$ for all x such that $h(x)$ is in I.

Theorem 9.5. *p. 680* ***Differentiating and Integrating Power Series***

Suppose the power series

$$\sum_{k=0}^{\infty} c_k\,(x-a)^k,$$

converges for all $|x-a| < R$ and defines a function $f(x)$ on that interval.

1. **Differentiation of Power Series**

   The $f(x)$ is differentiable (and thus continuous) for all $|x-a| < R$. Moreover, we can find the derivative $f'(x)$ by differentiating the power series for f term by term

$$f'(x) = \frac{d}{dx}\left[\sum_{k=0}^{\infty} c_k\,(x-a)^k\right]$$

$$= \sum_{k=0}^{\infty} c_k\,\frac{d}{dx}\left[\,(x-a)^k\,\right]$$

$$= \sum_{k=0}^{\infty} k\,c_k\,(x-a)^k.$$

   for $|x-a| < R$.

2. **Integration of Power Series**

   The indefinite integral of f is found by integrating the power series for f, term by term with

$$\int f(x)\,dx = \int \left[\sum_{k=0}^{\infty} c_k\,(x-a)^k\right] dx$$

$$= \sum_{k=0}^{\infty} c_k \int \left[\,(x-a)^k\,\right] dx$$

$$= \sum_{k=0}^{\infty} c_k\,\frac{(x-a)^{k+1}}{k+1} + c.$$

   for $|x-a| < R$, where c is an arbitrary constant.