Lesson 22: Taylor Series
Reference: Brigg's "Calculus: Early Transcendentals, Second Edition"
Section 9.3: Taylor Series, p. 684 - 696

Definition. *p. 685* ***Taylor Series for a function***

Let $f(x)$ be a single-variable function with continuous derivatives of all orders on an interval centered at the point $x = a$. The **Taylor series for f centered at point a** is

$$f(x) = f(a) + f'(a)\,(x-a) + \frac{f''(a)}{2!}\,(x-a)^2 + \frac{f^{(3)}(a)}{3!}\,(x-a)^3 + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}\,(x-a)^k.$$

Remark: For a Taylor series representation of a function to be useful, we need to know:

A. The values of x for which the Taylor series converges

B. The values for x for which the output of the Taylor series representation *equals* f

Definition. *p. 685* ***Maclaurin Series for a function***

Let $f(x)$ be a single-variable function with continuous derivatives of all orders on an interval centered at the point $x = 0$. The **Maclaurin series for f** is the Taylor series for f centered at $a = 0$ and given by

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \frac{f^{(3)}(0)}{3!}\,x^3 + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}\,x^k.$$

Definition. *p. 688* ***Nonnegative Integer Binomial Coefficients***

For nonnegative integers $p, k \in \mathbb{Z}$ with $0 \leq k \leq p$, we define the **binomial coefficients** to be given by

$$\binom{p}{k} = \frac{p!}{k!\,(p-k)!}$$

These coefficients for the rows of Pascal's Triangle.

Polynomial Tool Box Pascal's Triangle using Binomial Coefficients

$$\begin{array}{lccccccccccc}
n=0: & & & & & & \binom{0}{0} & & & & & \\
n=1: & & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
n=2: & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
n=3: & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
n=4: & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
n=5: & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}$$

Polynomial Tool Box Pascal's Triangle (Calculated Values)

$$\begin{array}{lccccccccccc}
n=0: & & & & & & 1 & & & & & \\
n=1: & & & & & 1 & & 1 & & & & \\
n=2: & & & & 1 & & 2 & & 1 & & & \\
n=3: & & & 1 & & 3 & & 3 & & 1 & & \\
n=4: & & 1 & & 4 & & 6 & & 4 & & 1 & \\
n=5: & 1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

Theorem. ***Binomial Theorem for perfect positive powers***

For nonnegative integers $p \in \mathbb{Z}$, the polynomial

$$(x+a)^p = \sum_{k=0}^{p} \binom{p}{k} x^{p-k} a^k$$

Polynomial Tool Box **First few cases of Binomial Theorem**

Binomial Theorem	Example
$(x+a)^2 = x^2 + 2ax + a^2$	$(x+1)^2 = x^2 + 2x + 1$
$(x-a)^2 = x^2 - 2ax + a^2$	$(x-1)^2 = x^2 - 2x + 1$
$(x+a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$	$(x+1)^3 = x^3 + 3x^2 + 3x + 1$
$(x-a)^3 = x^3 - 3ax^2 + 3a^2x - a^3$	$(x-1)^3 = x^3 - 3x^2 + 3x + 1$
$(x+a)^4 = x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4$	$(x+1)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1$
$(x-a)^4 = x^4 - 4ax^3 + 6a^2x^2 - 4a^3x + a^4$	$(x-1)^4 = x^4 - 12x^3 + 54x^2 - 108x + 81$

Definition. *p. 688* ***General Binomial Coefficients***

For real numbers $p \in \mathbb{R}$ and natural numbers $k \in \mathbb{N}$, we define the **binomial coefficents** to be given by

$$\binom{p}{k} = \frac{p\,(p-1)\,(p-2)\cdots(p-k+1)}{k!}, \qquad \binom{p}{0} = 1.$$

Theorem 9.6. *p. 689* ***Binomial Series***

For real numbers $p \in \mathbb{R}$ not equal to zero, the Taylor series for function

$$f(x) = (1+x)^p$$

centered at 0 is the **binomial series** given by

$$\sum_{k=0}^{\infty} \binom{p}{k} x^k = 1 + \sum_{k=1}^{\infty} \frac{p\,(p-1)\,(p-2)\cdots(p-k+1)}{k!}\, x^k$$

$$= 1 + p\,x + \frac{p\,(p-1)}{2!}\, x^2 + \frac{p\,(p-1)\,(p-2)}{3!}\, x^3 + \frac{p\,(p-1)\,(p-2)\,(p-3)}{4!}\, x^4 + \cdots.$$

The series converges for $|x| < 1$ (and possibly at the endpoints, depending on p). If p is a nonnegative integer, the series terminates and results in a polynomial of degree p.

Theorem 9.7. *p. 692* ***Convergence of Taylor Series***

Let $f(x)$ be a function with continuous derivatives of all orders on an open interval $I \subseteq \mathbb{R}$ that contains constant a. The Taylor series for f centered at a converges to f, for all $x \in I$, if and only if

$$\lim_{n \to \infty} R_n(x) = 0$$

for all x in I, where the remainder function at x is given by

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}\,(x-a)^{n+1}$$

where c is a properly chosen point between x and a.