

Example 2: Linear Algebraic Nodal Analysis Algorithm
 Circuit with 8 Resistors, 2 Voltage Sources, 2 Current Sources
 (Revised version of Example 4.2.1 p. 122 of Artice Davis's Textbook)

Example 2: Ideal schematic diagram

In this example, we build an electric circuit that includes eight $1\text{k}\Omega$ resistors, two 5V dc voltage sources, and two 2.5mA dc current sources. Figure 1 is an ideal circuit diagram that provides a complete description of this circuit. Notice that in this diagram, we label and enumerate each ideal circuit element and assign associated values.

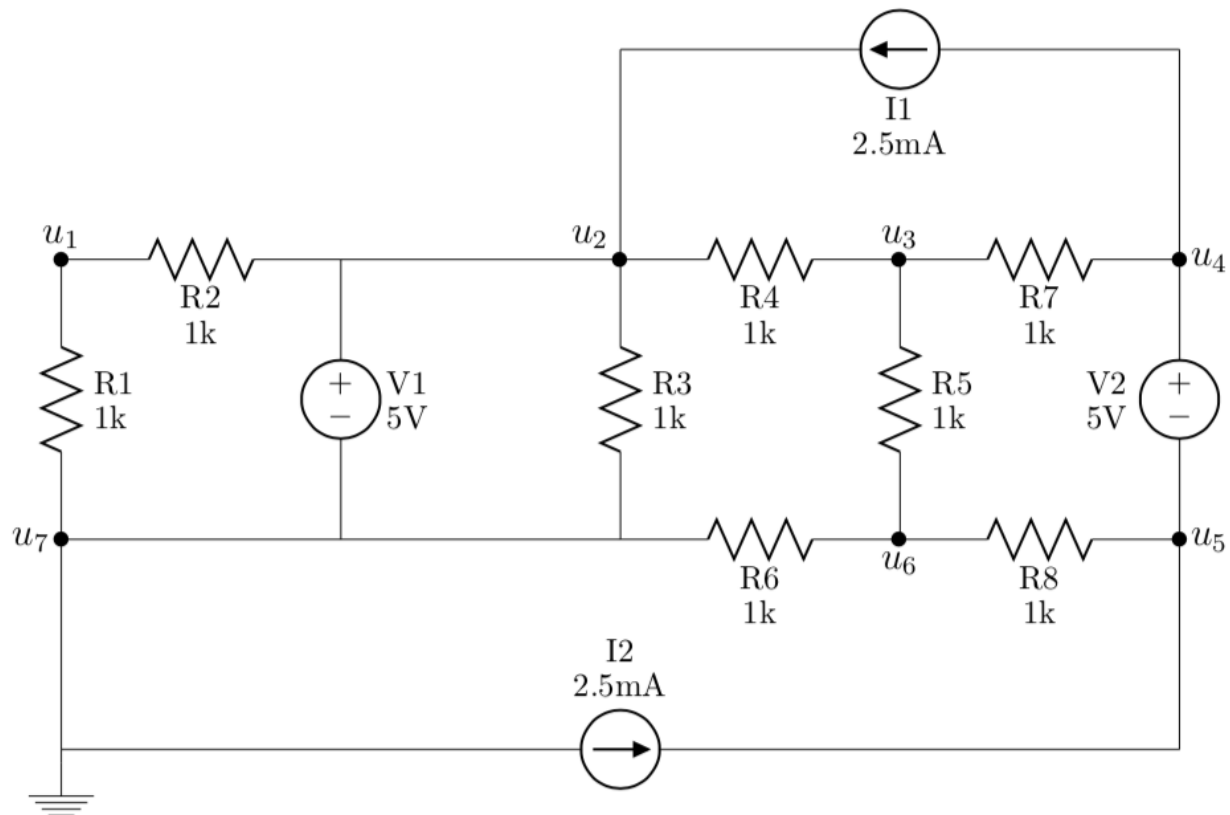


Figure 1: Ideal circuit schematic diagram

Example 2: Parts needed for this circuit

To build this circuit, we need the following parts:

- | | |
|--|---|
| <input type="checkbox"/> One half-size 2.2"x 3.4" solderless breadboard | <input type="checkbox"/> Two 5V dc voltage sources |
| <input type="checkbox"/> Two red 0.2" jumper wires | <input type="checkbox"/> Two 2.5mA dc current sources |
| <input type="checkbox"/> One blue 0.6" jumper wire | <input type="checkbox"/> Four fully-charged 9V batteries |
| <input type="checkbox"/> Eight 1k resistors (brn-brn-blk-blk-brn) | |

Example 2: Get started with a picture of the physical circuit

In Figure 2, we see a wiring diagram for the circuit we build in this example. The reader can find detailed instructions on how to prototype this circuit at the bottom of this page under the “Example 2: Build this circuit” section.

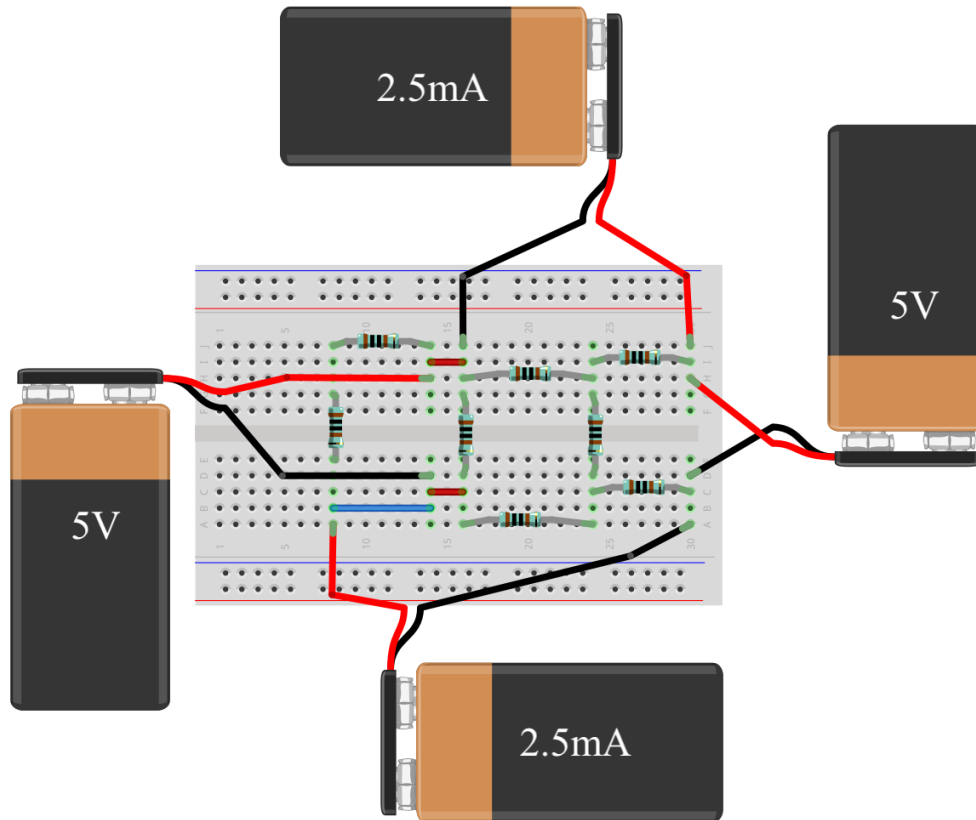


Figure 2: Digital Wiring Diagram for the Real Circuit.

Example 2: Build this circuit

To build the circuit, please grab all the necessary parts described in the “Example 2: Parts needed for this circuit” section on the first page of this document. Then, follow the instructions below:

1. Disconnect batteries from dc sources
2. Insert red 0.2" jumper across i14 and i16
3. Insert red 0.2" jumper across c14 and c16
4. Insert blue 0.6" jumper across b8 and b14
5. Insert resistor r_1 across d8 and g8
6. Insert resistor r_2 across j8 and j14
7. Insert resistor r_3 across d16 and g16
8. Insert resistor r_4 across h16 and h24
9. Insert resistor r_5 across d24 and g24
10. Insert resistor r_6 across a16 and a24
11. Insert resistor r_7 across i24 and i30
12. Insert resistor r_8 across b24 and b30
13. Insert red lead of v_{s_1} in i14 and blk lead in d14
14. Insert red lead of v_{s_2} in h30 and blk lead in c30
15. Insert red lead of i_{s_1} in j30 and blk lead in j16
16. Insert red lead of i_{s_2} in a8 and blk lead in a30

Example 2: Measure the node voltage potentials

Using the digital multimeter, let's measure the voltage potential at the seven nodes of our circuit. We begin by connecting the negative lead of our multimeter to node seven. Then, we connect the positive lead of the multimeter to each of the seven nodes, one by one, to capture the voltage meter. In figure 3, we enumerate the node locations on the wiring diagram for this circuit. In figure 3B, we should how to measure the voltage potential at node 1.

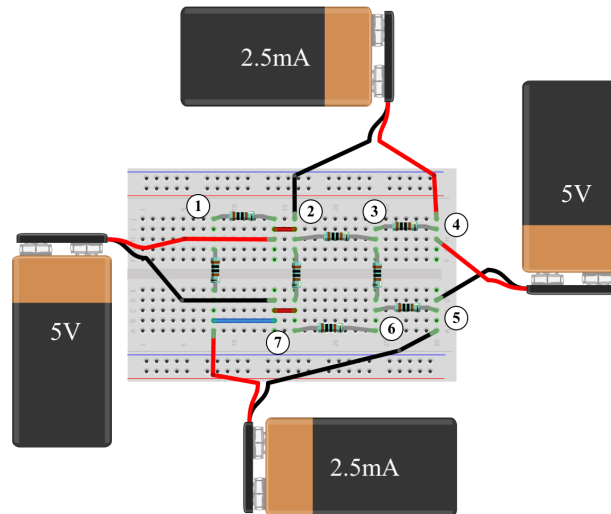


Figure 3: Node Labels on Wiring Diagram.

As we capture this data, we fill out the table below by hand. In order to assist readers in judging the accuracy of voltage measurements for this circuit example, we provide a YouTube video of one experimental data set using this example circuit.

LANA Example 2, Table 1: Node potential measurements

Node	Node Variable	Measured value (V)	Modeled value (V)
1	u_1		
2	u_2		
3	u_3		
4	u_4		
5	u_5		
6	u_6		
7	u_7		

Compare and contrast the ideal schematic and wiring diagram

Let's compare and contrast the ideal diagram with the wiring diagram. Notice that each ideal circuit element seen in Figure 1 corresponds to a unique real circuit element pictured in Figure 2. Using this observation, please make a bijection between the ideal circuit elements and the corresponding physical components used to build the real circuit. By doing so, we train ourselves to translate back and forth between an ideal circuit schematic diagram and the corresponding wiring diagram. This is valuable skill and takes practice.

In the work that follows, we analyze the electrical behavior of this circuit using the linear-algebraic nodal analysis (LANA) algorithm. Interested readers can download a full description of this algorithm on the support website for this work, provided at the URL below:

<http://www.appliedlinearalgebra.com/blog/for-teachers/linear-algebra-laboratory-exercises/electrify-linear-systems>

On that website, readers can click on the link titled "The Linear Algebraic Nodal Analysis Algorithm (.pdf)." This will open a two-page document with a detailed overview of each step of the LANA algorithm. For a more thorough description of this work, readers might consult the paper *Electrify the linear systems problem for our students* (CITE). A soft copy of this paper is also available on the same support website mentioned above. Let's begin analyzing our circuit with the first step of the LANA algorithm.

Step 1: Identify and label the entire set of nodes in our circuit

We start our work in the LANA algorithm by explicitly identifying and enumerating the individual nodes of our ideal circuit. To do so, we use the *node identification heuristic* in which we erase the element bodies of all ideal circuit elements and leave only the leads. The remaining, contiguous segments of conductor are known as the *nodes* of the circuit. In Figure 3 below, we redraw our ideal circuit schematic diagram from in Figure 1 using this node identification heuristic. We then count and enumerate the nodes of this circuit using positive integers. In this example, we see our circuit has seven nodes.

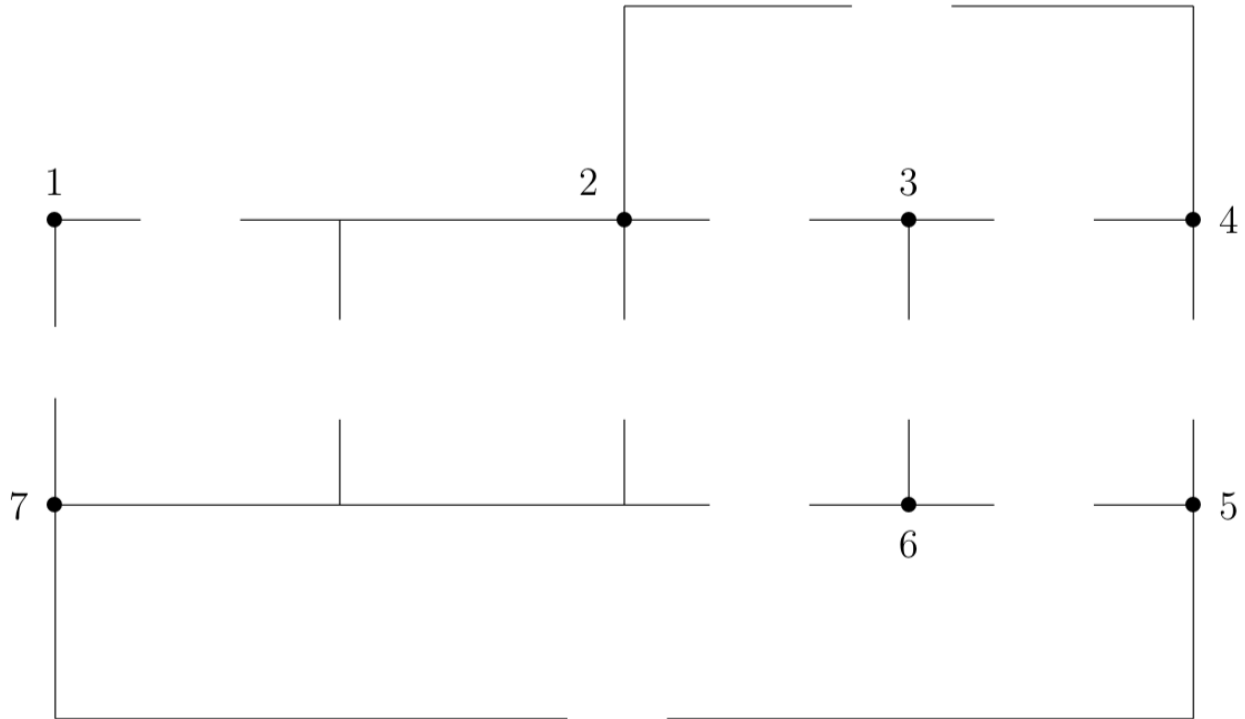


Figure 3: Node identification heuristic to identify and enumerate the nodes of circuit.

Step 2: Model the circuit using a directed graph

The second step in the LANA algorithm is to represent the topology (FOOTNOTE) of our circuit as a directed graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$. We should, of course, check that we are given a complete description of a circuit that contains only resistors, dc voltage sources, and dc current sources. The circuit presented in Figure 1 of this example meets these specifications.

Step 2A: Track the dimensions of key features in our circuit

The nodes of our digraph correspond to the circuit nodes and represent the interconnections between the circuit elements. For this specific example, we see

$$\mathcal{N} = \{1, 2, 3, 4, 5, 6, 7\}.$$

The positive integer $n_g \in \mathbb{N}$ denotes the number of elements in our set of nodes \mathcal{N} . In this example, $n_g = 7$ since we identified and enumerated 7 nodes in step 1 of this algorithm.

Next, we replace each circuit element with a directed edge in our digraph model. We can write each edge $e \in \mathcal{E}$ as an ordered pair of nodes with $\mathcal{E} \subset \mathcal{N} \times \mathcal{N}$. The direction of edge $e = (2, 1)$ goes from node 2 to node 1. In other words, we say that edge $e = (2, 1)$ *leaves* node 2 and *enters* node 1.

The variable $m \in \mathbb{N}$ represents the total number of edges in our graph. Because each edge of our digraph corresponds to a single circuit element, the number of edges equals the number of circuit elements. To count the edges, we sum of the number of resistors, denoted as $m_r \in \mathbb{N}$, the number of dc voltage sources, denoted as a nonnegative integer $m_v \in \mathbb{Z}$, and the number of dc current sources, denoted as a nonnegative integer $m_i \in \mathbb{Z}$. Mathematically, we write

$$m = m_r + m_v + m_i.$$

As we see in Figures 1 and 2, there are a total of $m_r = 8$ resistors, $m_v = 2$ dc voltage sources, and $m_i = 2$ current sources. Accordingly, our directed graph model of this circuit has $m = 12$ edges.

Step 2B: Orient and enumerate the edges of the digraph

We choose the direction of each edge using some knowledge of electronics. Specifically, we orient each edge associated with a dc current source in the same direction as the current flow provided by said source. The direction of each edge corresponding to a dc voltage source points from the positive “+” lead to the negative “-” lead of that source. Finally, we assign arbitrary directions to all edges corresponding to current flowing through the resistors.

We should note that we choose a very special enumeration scheme for the edges of our circuit. First we count and label all edges corresponding to resistors as edges e_1, e_2, \dots, e_{m_r} . We continue our edge count by labeling the edges associated with the voltage sources, yielding edges $e_{m_r+1}, \dots, e_{m_r+m_v}$. Finally, we enumerate our edges corresponding to current sources, producing edges $e_{m_r+m_v+1}, \dots, e_m$.

The only condition required in each count is that we have a bijection between circuit elements and graph edges. Although no specific sub-ordering is necessary, we recommend counting from top to bottom, left to right whenever possible. By enumerating our edges using these guidelines, we enable a useful partition of the matrices and vectors that model our circuit. Figure 4 above combine our rules for drawing directed edges with the enumeration scheme to produce a visual representation of our set of edges. We can also write this choice of edges in set notation as

$$\mathcal{E} = \{ \underbrace{(1, 7)}_{e_1}, \underbrace{(2, 1)}_{e_2}, \underbrace{(2, 7)}_{e_3}, \underbrace{(2, 3)}_{e_4}, \underbrace{(3, 6)}_{e_5}, \underbrace{(6, 7)}_{e_6}, \underbrace{(3, 4)}_{e_7}, \underbrace{(5, 6)}_{e_8}, \underbrace{(2, 7)}_{e_9}, \underbrace{(4, 5)}_{e_{10}}, \underbrace{(4, 2)}_{e_{11}}, \underbrace{(7, 5)}_{e_{12}} \}.$$

Step 2C: Draw a directed graph model of the circuit.

Using the guidelines provided above, we draw a digraph model of our circuit, seen in Figure 4.

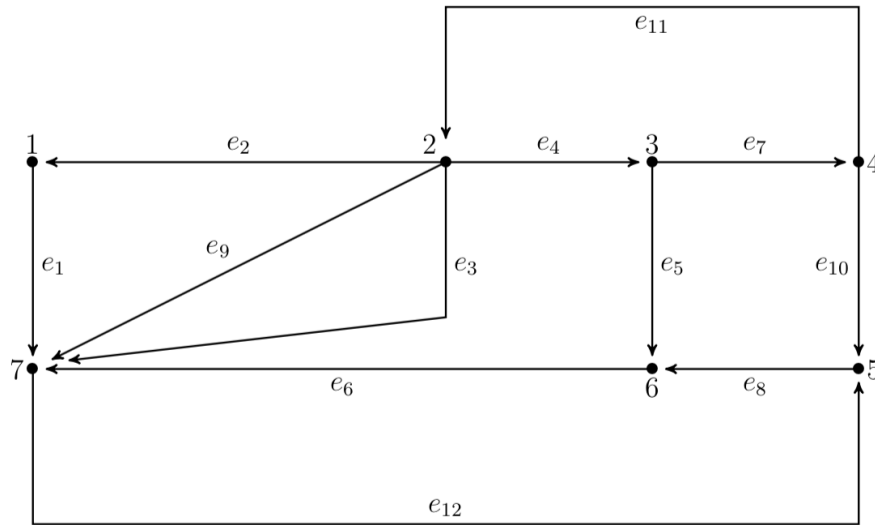


Figure 4: Digraph to model of our circuit in which each circuit element is replaced by a directed edge. In this diagram, we have labeled each node of the digraph labeled using an integer.

Step 3: Create the entire incidence matrix

Next, we construct a *entire incidence matrix* to model the connectivity of the nodes in this directed graph. The entry-by-entry definition of the entire incidence matrix $A_g \in \mathbb{R}^{m \times n_g}$ is given by

$$a_{jk} = \begin{cases} 1 & \text{if edge } e_j \text{ leaves node } k, \\ -1 & \text{if edge } e_j \text{ enters node } k, \\ 0 & \text{otherwise,} \end{cases}$$

for $j = 1, 2, \dots, m$ and $k = 1, 2, \dots, n_g$. We see that the rows and columns of this matrix correspond to the edges and the nodes of our digraph, respectively. Below we set up this entire incidence matrix table and explicitly label the rows and columns of this matrix:

		Nodes						
		1	2	3	4	5	6	7
Edges	e_1	1	0	0	0	0	0	-1
	e_2	-1	1	0	0	0	0	0
	e_3	0	1	0	0	0	0	-1
	e_4	0	1	-1	0	0	0	0
	e_5	0	0	1	0	0	-1	0
	e_6	0	0	0	0	0	1	-1
	e_7	0	0	1	-1	0	0	0
	e_8	0	0	0	0	1	-1	0
	e_9	0	1	0	0	0	0	-1
	e_{10}	0	0	0	1	-1	0	0
	e_{11}	0	-1	0	1	0	0	0
	e_{12}	0	0	0	0	-1	0	1

Table 1: Directed Incidence Table

We can transform this table into the block-partitioned entire incidence matrix

$$A_g = \begin{bmatrix} \vdots \\ A_{r_g} \\ \dots \\ A_{v_g} \\ \dots \\ A_{i_g} \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ \dots \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

where $A_{r_g} \in \mathbb{R}^{m_r \times n_g}$, $A_{v_g} \in \mathbb{R}^{m_v \times n_g}$, and $A_{i_g} \in \mathbb{R}^{m_i \times n_g}$. When defining the block partitions of any matrix or vector in our modeling scheme, we use subscripts r , v , and i to denote the subblocks corresponding to digraph edges that encode resistors, voltage sources, and current sources, respectively.

Step 4: Create all circuit vectors

The fourth step in our LANA algorithm is to describe the electrical behavior of each element in this circuit using variables that are easy to organize, refer to, and manipulate. For this purpose, we rely on matrix and vector arithmetic from applied linear algebra. In particular, we store all pertinent circuit variables in vector format. This structure facilitates our ability to state the set of key circuit equations in matrix form.

Step 4A: Define circuit variables for the node voltage potentials

From step 1 of our algorithm, we know there are $n_g = 7$ distinct nodes in this circuit. Thus, let's create a vector $\mathbf{u}_g \in \mathbb{R}^{n_g}$ of node voltage potentials given by

$$\mathbf{u}_g = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} \text{the voltage potential of node 1} \\ \text{the voltage potential of node 2} \\ \text{the voltage potential of node 3} \\ \text{the voltage potential of node 4} \\ \text{the voltage potential of node 5} \\ \text{the voltage potential of node 6} \\ \text{the voltage potential of node 7} \end{bmatrix}$$

As described below, in step 7 of the LANA algorithm we specifically choose a designate a *datum* node $d \in \mathcal{N}$, also called the *ground* node. Once we do so, we set the corresponding voltage potential $u_d = 0$. However, in this early stage of our modeling process, we include all node voltage potentials in the vector \mathbf{u}_g . The subscript g indicates that the ground node is included and has yet to be deleted.

Step 4B: Define circuit variables for the voltage drops across each element

Next, we create our vector $\mathbf{v} \in \mathbb{R}^m$ that stores the voltage drop across each element, with

$$\mathbf{v} = \begin{bmatrix} \mathbf{v}_r \\ \mathbf{v}_v \\ \mathbf{v}_i \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ \hline v_9 \\ v_{10} \\ \hline v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} \text{the voltage drop across resistor 1} \\ \text{the voltage drop across resistor 2} \\ \text{the voltage drop across resistor 3} \\ \text{the voltage drop across resistor 4} \\ \text{the voltage drop across resistor 5} \\ \text{the voltage drop across resistor 6} \\ \text{the voltage drop across resistor 7} \\ \text{the voltage drop across resistor 8} \\ \hline \text{the voltage drop across voltage source 1} \\ \text{the voltage drop across voltage source 2} \\ \hline \text{the voltage drop across current source 1} \\ \text{the voltage drop across current source 2} \end{bmatrix}$$

where $\mathbf{v}_r \in \mathbb{R}^{m_r}$, $\mathbf{v}_v \in \mathbb{R}^{m_v}$, and $\mathbf{v}_i \in \mathbb{R}^{m_i}$. Since we calculate one voltage drop across each element in our circuit and since there are precisely $m = 12$ elements in our circuit, we know there are exactly 12 entries in this vector. Due to our chosen strategy for enumerating the edges, the vector \mathbf{v} lists all voltage drops across the resistors followed by the voltage drops across the voltage sources and ending with voltage drops across the current sources. The stated partition of \mathbf{v} that includes subblocks $\mathbf{v}_r, \mathbf{v}_v, \mathbf{v}_i$ follows from this ordering.

Step 4C: Define circuit variables for the current running through each element

Finally, we create our vector $\mathbf{i} \in \mathbb{R}^m$ to store the current running through each element, given by

$$\mathbf{i} = \begin{bmatrix} \mathbf{i}_r \\ \mathbf{i}_v \\ \mathbf{i}_i \end{bmatrix} = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \\ i_7 \\ i_8 \\ \hline i_9 \\ i_{10} \\ \hline i_{11} \\ i_{12} \end{bmatrix} = \begin{bmatrix} \text{the current through resistor 1} \\ \text{the current through resistor 2} \\ \text{the current through resistor 3} \\ \text{the current through resistor 4} \\ \text{the current through resistor 5} \\ \text{the current through resistor 6} \\ \text{the current through resistor 7} \\ \text{the current through resistor 8} \\ \hline \text{the current through voltage source 1} \\ \text{the current through voltage source 2} \\ \hline \text{the current through current source 1} \\ \text{the current through current source 2} \end{bmatrix}$$

where $\mathbf{i}_r \in \mathbb{R}^{m_r}$, $\mathbf{i}_v \in \mathbb{R}^{m_v}$, and $\mathbf{i}_i \in \mathbb{R}^{m_i}$. Since we calculate one current through each element in our circuit and since there are precisely 12 branches in this example network, we know there are exactly 12 entries in this vector. Again, using the same convention, we list all currents through the resistors first, followed by the currents through the voltage sources and finally the currents through the current sources.

Optional Extension to Step 4: Label all variables and reference directions

One unnecessary but illuminating step in this process is to redraw the ideal circuit model and explicitly label all the voltage and current variables for every circuit element, as seen in Figure 5 below. Included in our revised ideal circuit schematic will be a specific indication of the reference directions for the voltage drop across each element as well as the reference direction for the current running through each element. In our case, we always draw the reference current direction with the initial point of the current vector starting at the lead with positive voltage potential and the terminal point of the current vector ending at the lead with negative voltage potential. In other words, we assume the passive sign convention in which the current arrow points from positive to negative (CITE: Artice Davis).

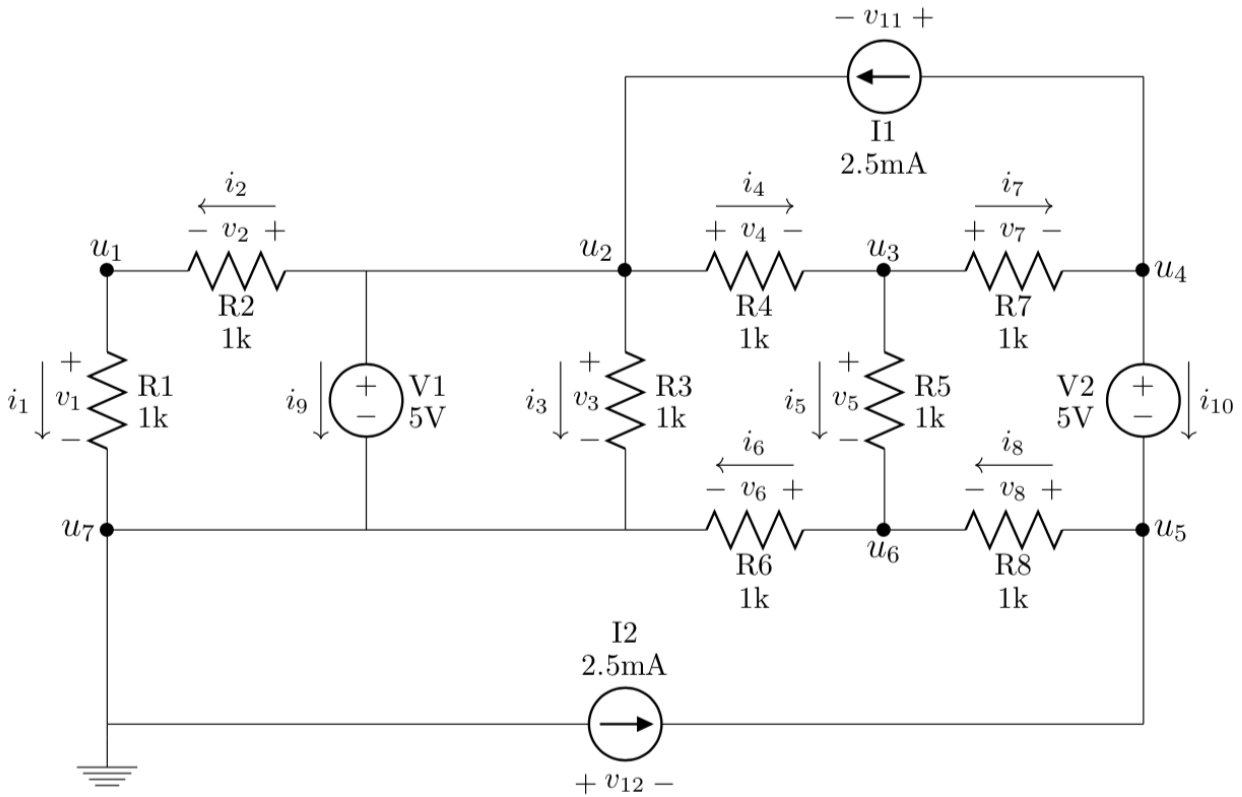


Figure 5: Redrawn Ideal Circuit Schematic Diagram with all variables labeled.

Notice that not all the scalar entries of the vectors $\mathbf{v}, \mathbf{i} \in \mathbb{R}^m$ are unknown variables. More specifically, the vectors \mathbf{v}_v and \mathbf{i}_i store known constants defined by the voltage and current levels provided by the independent sources we use to power our physical circuits. For example, we can use our assigned values for each dc voltage source provided in the ideal circuit schematic in Figure 1 to conclude that

$$\mathbf{v}_v = \begin{bmatrix} v_{v_1} \\ v_{v_2} \end{bmatrix} = \begin{bmatrix} v_9 \\ v_{10} \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}.$$

For $j = 1, \dots, m_v$, we say that v_{v_j} represents the assigned voltage value of the j th dc voltage source V_j in our circuit, where these constants are measured in volts. Using the same reasoning, we know that

$$\mathbf{i}_i = \begin{bmatrix} i_{i_1} \\ i_{i_2} \end{bmatrix} = \begin{bmatrix} i_{11} \\ i_{12} \end{bmatrix} = \begin{bmatrix} 2.5 \\ 2.5 \end{bmatrix}.$$

For $j = 1, \dots, m_i$, we say that i_{i_j} represents the assigned current running through the j th dc current source I_j in our circuit, as measured in milliamps.

Step 5: State the entire set of circuit equations

In step five of the LANA algorithm, we state the entire set of circuit equations which include the circuit KVLs, the circuit BCRs, and the circuit KCLs. Let's take a look at each of these equations. We also see how to combine these together to form a compact set of circuit equations that capture the electrical behavior of our circuit in a single matrix equation.

Step 5A: State entire set of Kirchoff's voltage laws in node potential form

We recall that our complete set of circuit equations begins with Kirchoff's voltage law in node potential form can be stated in block form as follows:

$$A_g \mathbf{u}_g = \mathbf{v} \quad \Leftrightarrow \quad \begin{bmatrix} A_{r_g} \mathbf{u}_g \\ A_{v_g} \mathbf{u}_g \\ A_{i_g} \mathbf{u}_g \end{bmatrix} = \begin{bmatrix} \mathbf{v}_r \\ \mathbf{v}_v \\ \mathbf{v}_i \end{bmatrix}$$

Using the subblock partitioning of the incidence matrix A_g and the voltage drop vector, we state the resulting element-specific KVLs. We begin with the *entire set of resistor KVLs*, given by

$$A_{r_g} \mathbf{u}_g = \mathbf{v}_r \quad \Leftrightarrow \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{bmatrix} \quad (1)$$

Notice that we can write the second subblock equation as a general linear-systems problem with known right-hand side vector. We call this equation the *entire set of voltage source KVLs in node potential form* and state these in matrix form as

$$A_{v_g} \mathbf{u}_g = \mathbf{v}_v \quad \Leftrightarrow \quad \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} v_9 \\ v_{10} \end{bmatrix} = \begin{bmatrix} v_{v_1} \\ v_{v_2} \end{bmatrix}. \quad (2)$$

We return to this general linear-system problem in steps 8, 9, and 10 of our algorithm. Finally, the *entire set of current-source KVLs* are given by the subblock equation

$$A_{i_g} \mathbf{u}_g = \mathbf{v}_i \quad \Leftrightarrow \quad \begin{bmatrix} 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix}. \quad (3)$$

Step 5B: State entire set of circuit branch constituent relations (BCRs)

Next, we state our branch constituent relations for the resistors in our circuit. In our case, we state the matrix version of Ohm's law given by the equation

$$\mathbf{v}_r = R \cdot \mathbf{i}_r \quad \Leftrightarrow \quad \mathbf{i}_r = G \cdot \mathbf{v}_r$$

where $R \in \mathbb{R}^{m_r \times m_r}$ is a diagonal matrix with the appropriate resistance values in each diagonal entry so that $v_k = r_k \cdot i_k$ for $k = 1, 2, \dots, m_r$. Since the k th resistor Rk has positive resistance value $r_k > 0$, we can also rewrite our Ohm's law equations in conductance form, where the conductance matrix $G = R^{-1}$ is a diagonal and the main diagonal entries are defined by the conductances $g_k = 1/r_k$. In our physical circuit, we have

$$\mathbf{i}_r = G \mathbf{v}_r \quad \Leftrightarrow \quad \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \\ i_7 \\ i_8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{bmatrix} \quad (4)$$

Step 5C: State entire set of Kirchoff's current laws

Finally we state the entire set of KCLs using the equation

$$A_g^T \mathbf{i} = \mathbf{0} \quad \Leftrightarrow \quad [A_{r_g}^T \mid A_{v_g}^T \mid A_{i_g}^T] \begin{bmatrix} \mathbf{i}_r \\ \mathbf{i}_v \\ \mathbf{i}_i \end{bmatrix} = \mathbf{0} \quad (5)$$

Since both A_g and \mathbf{i} are block partitioned matrices, we can restate our KCL's as

$$A_{r_g}^T \cdot \mathbf{i}_r + A_{v_g}^T \cdot \mathbf{i}_v + A_{i_g}^T \cdot \mathbf{i}_i = \mathbf{0} \quad (6)$$

We can state this equation in entry-by-entry form using the appropriate substitutions, yielding the equivalent equation

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 \\ -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \\ i_7 \\ i_8 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} i_9 \\ i_{10} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i_{11} \\ i_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (7)$$

Each row of this equation encodes the sum of all currents the enter and leave the corresponding node. Indeed, the entire incidence matrix A_g and each column of A_g corresponds to the nodes of our circuit. These KCL equations are written using the transpose of A_g . Since the transpose operation swaps rows and columns, it makes sense that the rows of the KCL equations also correspond to the nodes of our circuit.

Step 6: Determine the ordinary and generalized nodes of the circuit

Our next step is to determine the ordinary and generalized nodes of the circuit. We can do this by deactivating the circuit. To do so, we replace each current source with an open circuit by simply erasing the element bodies for each current source. On the other hand, we replace each voltage source with a short circuit by drawing each voltage source as a single lead.

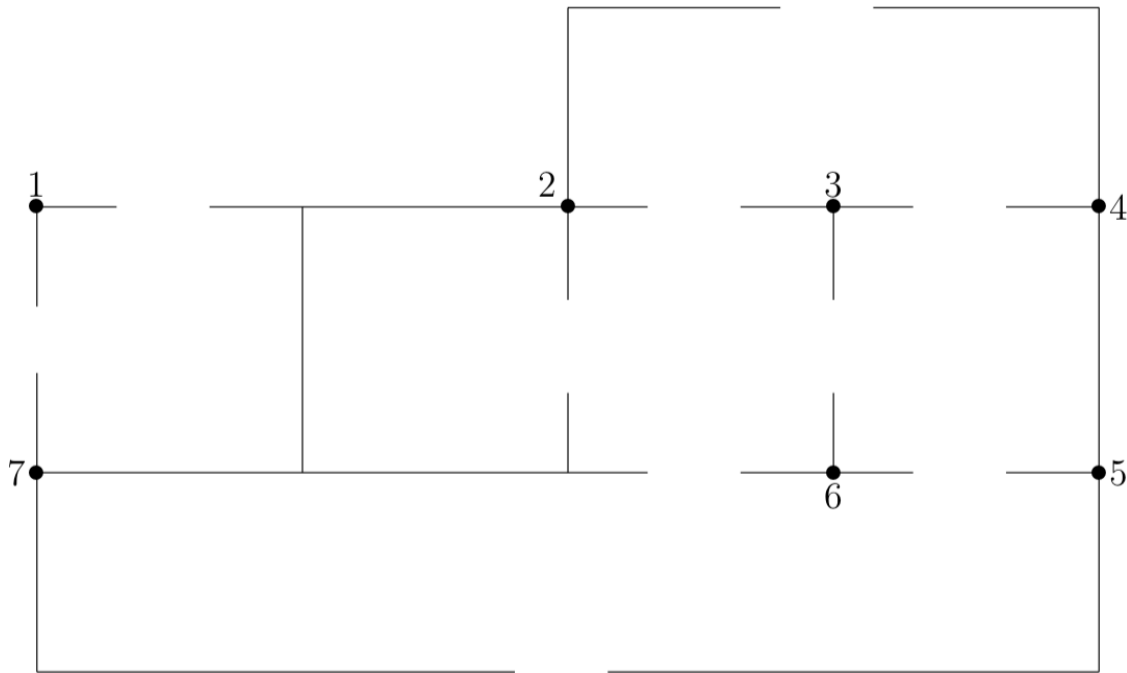


Figure 6: Redrawn node identification heuristic diagram for deactivated circuit.

Each *generalized node* in our system is actually a set of individual circuit nodes that we identified in Step 1 of our algorithm and that are connected by dc voltage sources. On the other hand, we say that an *ordinary node* of our system is a single node that we identified in Step 1 of our algorithm and to which no voltage source is connected. In step 8 of the LANA algorithm, we use these node categories to strategically reduce our system of equations. For now, our goal is to classify every node in our system as either belonging to a generalized node or as an ordinary node. For the specific circuit seen in Figure 1 of this example, our node classification scheme is provided in Table II below.

Type of node	Set of node indices	Node variables	Cardinality
Ordinary nodes	{1, 3, 6}	u_1, u_3, u_6	3
Generalized node 1	{4, 5}	u_4, u_5	2
Generalized node 2	{2, 7}	u_2, u_7	2

Table II: Classification of ordinary and generalized nodes

Step 7: Ground the circuit

Step 7A: Choose a datum node for the circuit

In this case, let's choose our datum node to be node $d = 7$. Using the column-partition definition of matrix-vector multiplication, we can revisit the *entrie set of resistor KVLs*, given by

$$\mathbf{v} = A_g \mathbf{u}_g = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ \hline 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix}$$

$$= u_1 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + u_2 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + u_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 1 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + u_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ \hline 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + u_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ \hline 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + u_6 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ \hline 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + u_7 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ \hline -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Since we choose node 7 as our ground node, we set $u_7 = 0$ and the entire seventh column of this matrix disappears from our KVL equations. Thus, to express the grounded circuit equations, we have no need to mention of the seventh column nor do we need to refer to the entry u_7 once we've grounded our circuit. With this observation in mind, we define the *ground node deflation matrix*

$$D_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We then create the *grounded voltage potential vector*

$$\mathbf{u}_0 = D_0^T \mathbf{u}_g$$

We specify the entry-by-entry definition of this product as follows

$$\underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix}}_{\mathbf{u}_0} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{D_0^T} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix}}_{\mathbf{u}_g} \quad (8)$$

Step 7B: Ground the circuit equations using deflation

We then form the grounded incidence matrix A_0 by deflating the seventh column of the raw incidence matrix $A_g \in \mathbb{R}^{m \times n_g}$ to form the initially reduced incidence matrix $A_0 \in \mathbb{R}^{m \times n_0}$, with $A_0 = A_g D_0$. We can write this equation entry-by-entry as follows

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ \hline 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}}_{A_0} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ \hline 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}}_{A_g} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{D_0}$$

Notice that $A_0 \in \mathbb{R}^{m \times n_0}$ where $m = 12$ and $n_0 = (n_g - 1) = 6$. This reduction results from deleting the chosen datum node from our list of node voltage potentials. We can also update define component-specific subblocks of our grounded incidence matrix using matrix algebra, with

$$A_0 = \begin{bmatrix} A_{r_0} \\ \hline A_{v_0} \\ \hline A_{i_0} \end{bmatrix} = \begin{bmatrix} A_{r_g} \cdot D_0 \\ \hline A_{v_g} \cdot D_0 \\ \hline A_{i_g} \cdot D_0 \end{bmatrix} = A_g \cdot D_0,$$

We can focus on each subblock equation separately. Let's begin by calculating the *resistor subblock of the grounded incidence matrix* $A_{r_0} \in \mathbb{R}^{m_r \times n_0}$, where

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}}_{A_{r_0}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix}}_{A_{r_g}} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{D_0} \tag{9}$$

We continue by finding the *voltage-source subblock of the grounded incidence matrix* $A_{v_0} \in \mathbb{R}^{m_v \times n_0}$, where

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix}}_{A_{v_0}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix}}_{A_{v_g}} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{D_0} \tag{10}$$

Finally, we calculate the *current-source subblock of the grounded incidence matrix* $A_{i_0} \in \mathbb{R}^{m_i \times n_0}$, where

$$\underbrace{\begin{bmatrix} 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}}_{A_{i_0}} = \underbrace{\begin{bmatrix} 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}}_{A_{i_g}} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{D_0} \tag{11}$$

Step 8: State grounded circuit equations

Step 8A: State grounded KVLs in node potential form

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ \hline v_9 \\ v_{10} \\ \hline v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & -1 & 0 \\ \hline 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} = A_0 \cdot \mathbf{u}_0$$

This results in an equivalent set of *grounded KVL equations in node potential form* for the voltage drops across each element of the circuit, given by

$$\mathbf{v} = \begin{bmatrix} \mathbf{v}_r \\ \hline \mathbf{v}_v \\ \hline \mathbf{v}_i \end{bmatrix} = \begin{bmatrix} A_{r_0} \cdot \mathbf{u}_0 \\ \hline A_{v_0} \cdot \mathbf{u}_0 \\ \hline A_{i_0} \cdot \mathbf{u}_0 \end{bmatrix} = A_0 \cdot \mathbf{u}_0$$

Notice that this voltage drop matrix equation results in three subblock matrix equations. We begin with the *grounded KVLs for the resistors*, given by

$$A_{r_0} \mathbf{u}_0 = \mathbf{v}_r \quad \Leftrightarrow \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{bmatrix} \quad (12)$$

The second subblock equation, referred to as the *grounded KVLs for the voltage sources*, is given in entry-by-entry form as

$$A_{v_0} \mathbf{u}_0 = \mathbf{v}_v \quad \Leftrightarrow \quad \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} = \begin{bmatrix} v_9 \\ v_{10} \end{bmatrix} = \begin{bmatrix} v_{v_1} \\ v_{v_2} \end{bmatrix}. \quad (13)$$

Finally, the *grounded KVL equations for the current source* are given by the subblock equation

$$A_{i_0} \mathbf{u}_0 = \mathbf{v}_i \quad \Leftrightarrow \quad \begin{bmatrix} 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} = \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix}. \quad (14)$$

Step 8B: Recall entire set of circuit BCRs and combine with grounded KVLs

From our work in Step 5B of the LANA algorithm, we know that $\mathbf{i}_r = G \cdot \mathbf{v}_r$. We can combine this entire set of Ohm's law equations in conductance form with our grounded KVL equations for the resistor subblock of our circuit to see that

$$\mathbf{i}_r = G \cdot A_{r_0} \cdot \mathbf{u}_0. \quad (15)$$

Step 8C: State grounded KCLs

We can also write the grounded version of the Kirchoff's current law equations for our system with

$$A_0^T \cdot \mathbf{i} = \mathbf{0}.$$

Once again, since both A_0 and \mathbf{i} are block partitioned matrices, we can restate our KCL's as

$$A_{r_0}^T \cdot \mathbf{i}_r + A_{v_0}^T \cdot \mathbf{i}_v + A_{i_0}^T \cdot \mathbf{i}_i = \mathbf{0} \quad (16)$$

Step 8D: Write modified nodal analysis matrix equation

Using Step 8B of our algorithm, we can rewrite our grounded KCL equations in subblock form (16) by replacing the vector \mathbf{i}_r with the equivalent node potential equation (15), yielding

$$A_{r_0}^T \cdot G \cdot A_{r_0} \cdot \mathbf{u}_0 + A_{v_0}^T \cdot \mathbf{i}_v + A_{i_0}^T \cdot \mathbf{i}_i = \mathbf{0} \quad (17)$$

As mentioned in the Step 4 of our algorithm, the entries of the vector \mathbf{i}_i are known constants and thus we can bring the term $A_{i_0}^T \cdot \mathbf{i}_i$ in our grounded KCL equations in subblock form (16) to the right-hand side of the equals sign. This yields an equivalent system of grounded KCL equations where the contributions of the dc current sources are now expressed on the right-hand side, with

$$A_{r_0}^T \cdot G \cdot A_{r_0} \cdot \mathbf{u}_0 + A_{v_0}^T \cdot \mathbf{i}_v = -A_{i_0}^T \cdot \mathbf{i}_i. \quad (18)$$

This is reminiscent of the well-known approach we take to discuss forced harmonic oscillators in which we right the forcing function on the right-hand side of our governing equations. We have now properly accounted for the effect of choosing a ground node.

We are now in a position to use matrix algebra quite creatively. Let's look back at two important matrix subblock equations. First, let's recall the equation (13) to represent the voltage source contributions as a general linear-systems problem. We also recall the grounded KCL equations (18) where we write the contributions of the dc current source on the right-hand side. If we combine these two equations together and write the result in matrix form, we produce a famous linear-systems problem in the form

$$\begin{bmatrix} A_{r_0}^T \cdot G \cdot A_{r_0} & A_{v_0}^T \\ A_{v_0} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_0 \\ \mathbf{i}_v \end{bmatrix} = \begin{bmatrix} -A_{i_0}^T \cdot \mathbf{i}_i \\ \mathbf{v}_v \end{bmatrix}.$$

This problem was first introduced into the literature under the name modified nodal analysis algorithm from 1975 (CITE). In fact, many practicing engineers might stop here since this linear-systems problem can be solved using a (sparse) LU factorization algorithm resulting in values for our desired vectors \mathbf{u}_0 and \mathbf{i}_v . Another important note is that the modified nodal analysis algorithm enables us to handle much more general systems than the ones considered in this paper. For example, by using this blocked partition matrix, we can analyze systems with dependent current sources. However, as discussed before, the goal of this work is to empower students in our classes to create a nonsingular linear-systems problem with a positive definite

coefficient matrix. With a little bit of work, this linear systems problem can be written using a Gram matrix in the form

$$A^T \cdot C \cdot A.$$

This is the same matrix structure that Strang describes so well (CITE). Moreover, there are interesting connections between this coefficient matrix from modified nodal analysis and the corresponding matrices that arise in conventional nodal analysis as described by Artice Davis. To see these connections and generalize classical nodal analysis, we rely heavily on linear algebra.

Step 9: Identify essential nodes, nonessential nodes, and supernodes

Let's push our analysis of this system further. Remember that we say that the datum node is our chosen ground node. If the datum node is chosen within a generalized node, we set this node potential to zero and all other node potentials in this generalized node are known as *nonessential nodes*. We say that the *essential nodes* of our circuit are the ordinary, nondatum nodes. Each essential node yields a single, unknown, independent node voltage potential. Finally, we refer to any generalized node that does not contain the datum node as a *supernode*. For each supernode in our circuit, we need exactly one unknown, independent node voltage potential. All other nodes within each supernode can be calculated using the node potential of this independent node and the voltage source values within said node. Let's use this refinement to classify all nodes under the assumption that we choose node 7 as the datum node.

	Choose $u_7 = 0$		
	Variable	Set of indices	Cardinality
Datum node	u_7	$\mathcal{N}_d = \{7\}$	1
Nonessential node	u_2	$\mathcal{N}_n = \{2\}$	$n_n = 1$
Essential node	u_1, u_3, u_6	$\mathcal{N}_e = \{1, 3, 6\}$	$n_e = 3$
Supernode 1	u_4, u_5	$\mathcal{N}_{S_1} = \{4, 5\}$	$n_{s_1} = 2$
Supernode 2	None	$\mathcal{N}_{S_2} = \emptyset$	$n_{s_2} = 0$

Table III: Classification of essential nodes, nonessential nodes, and super nodes with choice of ground node 7

Step 10: Eliminate the node dependencies that arise due to the voltage sources

Let's begin by further classifying the nodes of our circuit. We look back at the entire set of voltage source KVL equations in node potential form (13). We can always transform these matrix equations into the corresponding scalar equations by delineating the entry-by-entry definitions of each voltage drop calculation with

$$A_{v_g} \mathbf{u}_g = \mathbf{v}_v \Leftrightarrow \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} v_{v_1} \\ v_{v_2} \end{bmatrix} \Leftrightarrow \begin{bmatrix} v_{v_1} \\ v_{v_2} \end{bmatrix} = \begin{bmatrix} u_2 - u_7 \\ u_4 - u_5 \end{bmatrix} \quad (19)$$

This is a general linear-systems problem with a known right-hand side vector \mathbf{v}_v . By a famous theorem in linear algebra, the general solution of this equation takes the form

$$\mathbf{u}_g = \mathbf{p}_g + D_v \mathbf{u}$$

where \mathbf{p}_g is a particular solution to our GLSP and the columns of D_v are related to the $\text{Nul}(A_{v_g})$. Below, we show how to form these quantities.

Step 10A: Determine the independent and dependent node voltage potentials

Since we chose node 7 as the datum node, we set $u_7 = 0$ and we see that node 2 is nonessential with known voltage value $u_2 = v_{v_1}$. We also know that each of the node voltage potentials u_1, u_3 , and u_6 at the essential nodes are independent. Finally, we can choose one independent node potential from our single supernode.

Step 10B: Create a general solution to the entire set of voltage-source KVL equations

- i. Suppose we say that u_4 is our independent node value. Then, we want to solve for u_5 in terms of u_4 . In this case, we see that $u_5 = u_4 - v_{v_2}$. Using this replacement, we can set up a general solution to the GLSP as follows

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} u_1 \\ v_{v_1} \\ u_3 \\ u_4 \\ u_4 - v_{v_2} \\ u_6 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ v_{v_1} \\ 0 \\ 0 \\ -v_{v_2} \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{p}_g} + \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{D_v} \underbrace{\begin{bmatrix} u_1 \\ u_3 \\ u_4 \\ u_6 \end{bmatrix}}_{\mathbf{u}}$$

- ii. Suppose, on the other hand, that we choose u_5 is our independent node value. Then, we solve for u_4 in terms of u_5 with $u_4 = u_5 + v_{v_2}$. Using this replacement, we can set up a general solution to the GLSP as follows

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} u_1 \\ v_{v_1} \\ u_3 \\ u_5 + v_{v_2} \\ u_5 \\ u_6 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ v_{v_1} \\ 0 \\ v_{v_2} \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{p}_g} + \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{D_v} \underbrace{\begin{bmatrix} u_1 \\ u_3 \\ u_5 \\ u_6 \end{bmatrix}}_{\mathbf{u}}$$

Step 10C: Create a general solution to the grounded set of voltage-source KVL equations

We can choose either options i. or ii. from Step 10B to rewrite the vector \mathbf{u}_g in it's general form for the solution of our GLSP equation (13). Let's assume we choose i. and select node 4 as our independent voltage source in the single supernode in this circuit. With this choice in mind, we recall that

$$\mathbf{u}_0 = D_0^T \mathbf{u}_g = D_0^T (\mathbf{p}_g + D_v \mathbf{u}) = \mathbf{p}_0 + D \mathbf{u} \quad (20)$$

where the $n_0 \times 1$ vector $\mathbf{p}_0 = D_0^T \mathbf{p}_g$ and the grounded voltage source deflation matrix $D \in \mathbb{R}^{n_0 \times n}$ is defined as $D = D_0^T D_v$. The entry-by-entry definitions of each of these matrices is given as

$$\underbrace{\begin{bmatrix} 0 \\ v_{v_1} \\ 0 \\ 0 \\ -v_{v_2} \\ 0 \end{bmatrix}}_{\mathbf{p}_0} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}}_{D_0^T} \underbrace{\begin{bmatrix} 0 \\ v_{v_1} \\ 0 \\ 0 \\ -v_{v_2} \\ 0 \end{bmatrix}}_{\mathbf{p}_g} \quad \text{and} \quad \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_D = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}}_{D_0^T} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{D_v}$$

Step 11: State the maximally deflated circuit equations

We are now in a position to make a fascinating claim. First, we notice that $A_{v_g} D_v = 0 \in \mathbb{R}^{m_v \times n}$ since

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix}}_{A_{v_g}} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{D_v} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Moreover, we claim that under modest assumptions that will likely be satisfied by all circuits used in practice, the columns of the matrix D form a basis for the null space of the matrix A_{v_0} with $A_{v_0} D = 0 \in \mathbb{R}^{m_v \times n}$. We confirm this fact here by noting

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix}}_{A_{v_0}} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{21}$$

Indeed, the matrix A_{v_0} has a total of six columns, two of which are clearly pivot columns and the other four of which must be nonpivot columns. But, the matrix D has four linearly independent columns in the null space of A_{v_0} and thus, by the fundamental theorem of linear algebra, must be a basis for this space.

Step 11A: Maximally deflate the incidence matrices

With this in mind, we can fully reduce our system. We recall our subblock equation (18) of the grounded KCLs with nonzero current-source forcing term on the right-hand side whter the resistor current vector is expressed in terms of the grounded node potential vector. Looking back at this equation, we now have an alternative expression for \mathbf{u}_0 and a deflation matrix D that annihilates A_{v_0} . Now we form our fully-deflated submatrices starting with the resistor subblock first with

$$A_r = A_{r_0} D \in \mathbb{R}^{m_r \times n} \Leftrightarrow \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}}_{A_r} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}}_{A_{r_0}} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_D$$

We've already shown that the columns of D send A_{v_0} to zero with $A_{v_0} D = 0$. Finally, we focus on deflating the current source subblock with

$$A_i = A_{i_0} D \in \mathbb{R}^{m_i \times n} \Leftrightarrow \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}}_{A_i} = \underbrace{\begin{bmatrix} 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}}_{A_{i_0}} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_D$$

Step 11B: Create the grounded and deactivated nodal equation in nonsingular form

We now use these matrices to simplify our circuit equations. If we multiply the compact KCL equation (18) on the left by the matrix D^T , we annihilate the term involving the voltage-source subblock $A_{v_0}^T$, yielding

$$A_r^T G A_{r_0} \mathbf{u}_0 = -A_i^T \mathbf{i}_i$$

Now we can substitute the expanded version of vector \mathbf{u}_0 into our equation to produce

$$A_r^T G A_{r_0} (\mathbf{p}_0 + D\mathbf{u}) = -A_i^T \mathbf{i}_i$$

If we set $\mathbf{f} = A_i^T \mathbf{i}_i$ and $\mathbf{b} = -A_{r_0} \mathbf{p}_0$, we can use this simplification to system as

$$A_r^T G A_r \mathbf{u} = A_r^T G \mathbf{b} - \mathbf{f}$$

where $K = A_r^T G A_r \in \mathbb{R}^{n \times n}$ and $\mathbf{f} \in \mathbb{R}^n$. This is now completely equilibrium equation framework espoused by Strang. Moreover, under this construction, the matrix A_r has full column rank. In particular, we note

$$\begin{aligned} K &= \underbrace{\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 0 \end{bmatrix}}_{A_r^T} \underbrace{\begin{bmatrix} g_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & g_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & g_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & g_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & g_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & g_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & g_7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_8 & 0 \end{bmatrix}}_G \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}}_{A_r} \\ &= \underbrace{\begin{bmatrix} g_1 & -g_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -g_4 & -g_5 & 0 & g_7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -g_7 & g_8 \\ 0 & 0 & 0 & 0 & -g_5 & g_6 & 0 & -g_8 \end{bmatrix}}_{A_r^T G} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}}_{A_r} \\ &= \underbrace{\begin{bmatrix} g_1 + g_2 & 0 & 0 & 0 \\ 0 & g_4 + g_5 + g_7 & -g_7 & -g_5 \\ 0 & -g_7 & g_7 + g_8 & -g_8 \\ 0 & -g_5 & -g_8 & g_5 + g_6 + g_8 \end{bmatrix}}_K \end{aligned}$$

The forcing terms on the right-hand side contain information about both the current sources and the effect of the voltage sources on the system. In particular, we notice that

$$\underbrace{\begin{bmatrix} 0 \\ 0 \\ i_{i_1} - i_{i_2} \\ 0 \end{bmatrix}}_{\mathbf{f}} = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & -1 \\ 0 & 0 \end{bmatrix}}_{A_i} \underbrace{\begin{bmatrix} i_{i_1} \\ i_{i_2} \end{bmatrix}}_{\mathbf{i}_i}$$

We also see that the voltage sources effect the system with

$$\underbrace{\begin{bmatrix} 0 \\ -v_{v_1} \\ -v_{v_1} \\ -v_{v_1} \\ 0 \\ 0 \\ 0 \\ v_{v_2} \end{bmatrix}}_{\mathbf{b}} = - \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}}_{A_{r_0}} \underbrace{\begin{bmatrix} 0 \\ v_{v_1} \\ 0 \\ 0 \\ -v_{v_2} \\ 0 \end{bmatrix}}_{\mathbf{p}_0}$$

To calculate the resulting currents, we see that

$$\underbrace{\begin{bmatrix} g_2 v_{v_1} \\ g_4 v_{v_1} \\ g_8 v_{v_2} \\ -g_8 v_{v_2} \end{bmatrix}}_{A_r^T G \mathbf{b}} = \underbrace{\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 \end{bmatrix}}_{A_r^T} \underbrace{\begin{bmatrix} g_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & g_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & g_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & g_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & g_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & g_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & g_7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_8 \end{bmatrix}}_G \underbrace{\begin{bmatrix} 0 \\ -v_{v_1} \\ -v_{v_1} \\ -v_{v_1} \\ 0 \\ 0 \\ 0 \\ v_{v_2} \end{bmatrix}}_{\mathbf{b}}$$

Combining all of these results together, we produce the same system as espoused by the classical nodal analysis algorithm with

$$\underbrace{\begin{bmatrix} g_1 + g_2 & 0 & 0 & 0 \\ 0 & g_4 + g_5 + g_7 & -g_7 & -g_5 \\ 0 & -g_7 & g_7 + g_8 & -g_8 \\ 0 & -g_5 & -g_8 & g_5 + g_6 + g_8 \end{bmatrix}}_K \underbrace{\begin{bmatrix} u_1 \\ u_3 \\ u_4 \\ u_6 \end{bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} g_2 v_{v_1} \\ g_4 v_{v_1} \\ g_8 v_{v_2} \\ -g_8 v_{v_2} \end{bmatrix}}_{A_r^T G \mathbf{b}} - \underbrace{\begin{bmatrix} 0 \\ 0 \\ i_{i_1} - i_{i_2} \\ 0 \end{bmatrix}}_{\mathbf{f}}$$

We can transform this equation into numerical form by substituting the exact values for that circuit components designated in figure 1 yielding the system

$$\underbrace{\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 3 \end{bmatrix}}_K \underbrace{\begin{bmatrix} u_1 \\ u_3 \\ u_4 \\ u_6 \end{bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} 5 \\ 5 \\ 5 \\ -5 \end{bmatrix}}_{A_r^T G \mathbf{b} - \mathbf{f}}$$

Step 12: Solve the linear-algebraic nodal analysis equations

Now we can solve our resulting 4×4 system using any method we'd like. In practice, I suggest using MATLAB since the whole point of this exercise is to set ourselves up to take advantage of computer algorithms. In fact, on the companion website for this paper, readers can find a MATLAB script to solve this exact problem. However, since this example write up is dedicated to learners who may have relatively little coding experience, we include the entire solution process for completeness and ease of reference. We begin by transforming our matrix into upper-triangular form using Gauss transformations:

$$\begin{aligned} \underbrace{\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 3 \end{bmatrix}}_K \underbrace{\begin{bmatrix} u_1 \\ u_3 \\ u_4 \\ u_6 \end{bmatrix}}_{\mathbf{u}} &= \underbrace{\begin{bmatrix} 5 \\ 5 \\ 5 \\ -5 \end{bmatrix}}_{A_r^T G \mathbf{b} - \mathbf{f}} \Leftrightarrow \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1/3 & 1 & 0 \\ 0 & 1/3 & 0 & 1 \end{bmatrix}}_{L_1} \underbrace{\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 3 \end{bmatrix}}_K \underbrace{\begin{bmatrix} u_1 \\ u_3 \\ u_4 \\ u_6 \end{bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1/3 & 1 & 0 \\ 0 & 1/3 & 0 & 1 \end{bmatrix}}_{L_1} \underbrace{\begin{bmatrix} 5 \\ 5 \\ 5 \\ -5 \end{bmatrix}}_{A_r^T G \mathbf{b} - \mathbf{f}} \\ \Leftrightarrow \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4/5 & 1 \end{bmatrix}}_{L_2} \underbrace{\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & -1 & -1 \\ 0 & 0 & 5/3 & -4/3 \\ 0 & 0 & -4/3 & 8/3 \end{bmatrix}}_{L_1 K} \underbrace{\begin{bmatrix} u_1 \\ u_3 \\ u_4 \\ u_6 \end{bmatrix}}_{\mathbf{u}} &= \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4/5 & 1 \end{bmatrix}}_{L_2} \underbrace{\begin{bmatrix} 5 \\ 5 \\ 20/3 \\ -10/3 \end{bmatrix}}_{L_1 (A_r^T G \mathbf{b} - \mathbf{f})} \\ \Leftrightarrow \underbrace{\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & -1 & -1 \\ 0 & 0 & 5/3 & -4/3 \\ 0 & 0 & 0 & 8/5 \end{bmatrix}}_{L_2 L_1 K} \underbrace{\begin{bmatrix} u_1 \\ u_3 \\ u_4 \\ u_6 \end{bmatrix}}_{\mathbf{u}} &= \underbrace{\begin{bmatrix} 5 \\ 5 \\ 20/3 \\ 2 \end{bmatrix}}_{L_2 L_1 (A_r^T G \mathbf{b} - \mathbf{f})} \end{aligned}$$

Using backward substitution, we see that

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_3 \\ u_4 \\ u_6 \end{bmatrix} = \begin{bmatrix} 2.50 \\ 3.75 \\ 5.00 \\ 1.25 \end{bmatrix}$$

Step 13: Completely describe electrical behavior of circuit

In this case, we choose to solve this problem using left multiplication by elementary matrices to transform our system into an equivalent linear-systems problem in upper triangular form. We then apply backward substitution to find the values of the independent voltage potentials stored in \mathbf{u} . Finally, we use our general linear-systems solution equation (20) to find all entries of the vector \mathbf{u}_0 .

Step 13A: Find all the node voltage potentials

Then, using our solution to the grounded linear-systems problem, we have

$$\underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix}}_{\mathbf{u}_0} = \underbrace{\begin{bmatrix} 0 \\ v_{v_1} \\ 0 \\ 0 \\ -v_{v_2} \\ 0 \end{bmatrix}}_{\mathbf{p}_0} + \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_D \underbrace{\begin{bmatrix} u_1 \\ u_3 \\ u_4 \\ u_6 \end{bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} 0 \\ 5 \\ 0 \\ 0 \\ -5 \\ 0 \end{bmatrix}}_{\mathbf{p}_0} + \underbrace{\begin{bmatrix} 2.50 \\ 0.00 \\ 3.75 \\ 5.00 \\ 5.00 \\ 1.25 \end{bmatrix}}_{D\mathbf{u}} = \begin{bmatrix} 2.50 \\ 5.00 \\ 3.75 \\ 5.00 \\ 0.00 \\ 1.25 \end{bmatrix}$$

Step 13B: Calculate any desired circuit variable values

One of the main uses of circuit analysis is to focus on a single circuit variable and report the value of this variable to any interested party when desired. The moment we solve for our given independent node voltage potentials, we claim that we can report the value of any variable using matrix-vector multiplication along with some trivial analysis. For example, suppose that we want to find the current flowing through any resistor in the entire circuit. We can do this immediately using the the matrix version of Ohm's law equation in conductance form with

$$\mathbf{i}_r = G\mathbf{v}_r.$$

The only other unknown quantities are the voltages across the current sources and the currents running through the voltage sources. The voltage drops across the current sources result from the current source subblock of the entire set of KVLs with

$$\mathbf{v}_i = A_{i_0}\mathbf{u}_0$$

On the other hand, the currents running through the voltage sources pose a slightly more subtle problem. But we have is our KCL equation

$$A_{v_0}^T \cdot \mathbf{i}_v = A_{r_0}^T \cdot G \cdot \mathbf{b} - A_{i_0}^T \cdot \mathbf{i}_i.$$

The left-hand side includes two unknown variables while the right-hand side is completely determined by our previously stated constants.

Step 14: Verify by measuring the node voltage potentials on the real circuit

Measure the node voltage potentials at all nodes in the circuit and record the data in the table below.

Node Variable	Measured value (V)	Modeled value (V)
u_1	2.497	2.50
u_2	4.98	5.00
u_3	3.728	3.75
u_4	4.95	5.00
u_5	0.003	0.00
u_6	1.241	1.25
u_7	0.000	0.00

Table III: Model verification

For practically-minded readers, a few notes about measurements are in order. If we were to measure the values of these variables on a physical circuit, we hold the black probe of the voltmeter to the ground node. Then we apply the red probe to each of the other nodes on the circuit. We would read the associated potentials using our voltmeter and record the data in a 7×1 vector. The $d = 7$ th entry of this vector is assumed to be zero since this is the reference voltage value against which all other potentials are measured.