For problems 1 - 4, let $f : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a two-variable function with explicit representation $z = f(x, y)$. Let $A\big(a, b, f(a, b)\big)$ be a point on the surface

$$S_f = \{(x, y, z) \, : \, (x, y) \in D \text{ and } z = f(x, y)\}.$$

Let $\mathbf{u} = \langle u_1, u_2 \rangle$ be a unit vector in the domain of function f.

1. (6 points) Please derive the limit definition of the directional derivative from first principles. If you're confused where to start, please follow the 5 steps process to constructing a derivative that we discussed in our Lesson 11 videos.

Solution: Recall the 5 step process for constructing derivative included each of the following:

I. Graph a curve C

In order to create the curve on which we will plot our tangent line, we begin with the graph of the surface defined by the explicit equation $z = f(x, y)$. To create the curve C along the surface, we restrict our input points in the domain to move along the line

$$\begin{aligned}\mathbf{r}(t) &= \mathbf{r}_0 + t \cdot \mathbf{u}, \\ &= \langle a, b \rangle + t \langle u_1, u_2 \rangle, \\ &= \langle a + tu_1, b + tu_2 \rangle, \\ &= \langle x(t), y(t) \rangle,\end{aligned}$$

where $x(h) = a + tu_1$ and $y(h) = b + tu_2$. This is equivalent to intersecting the surface $z = f(x, y)$ with a plane through the point $A\big(a, b, f(a, b)\big)$ with normal vector $\mathbf{n} = \langle -u_2, u_1, 0 \rangle$. This results in a single-variable function given by

$$g(t) = f(x(t), y(t)) = f\big(a + tu_1, b + tu_2\big).$$

Below, we visualize this curve and the points from step 2 of this process.

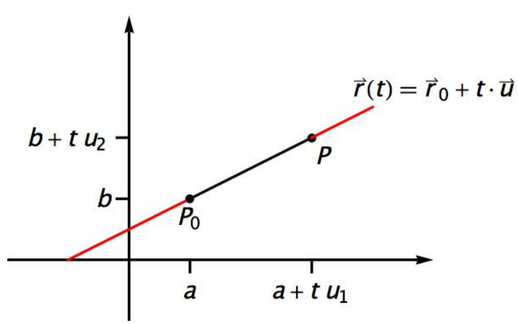


Figure 1A: Restricted domain inputs

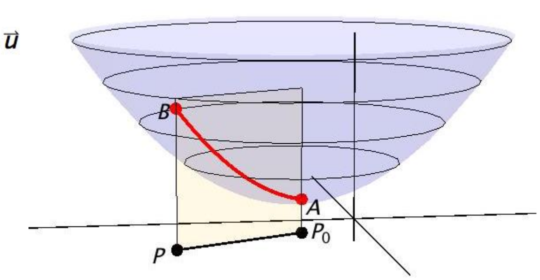


Figure 1B: Resulting curve on surface

Solution:

II. Find two points on the curve and draw a secant line between these two points.

We now find two points on the curve C. Since we will be finding the derivative at the point $A\big(a, b, f(a,b)\big)$, we start by noticing that the output value on the surface at this point is given by

$$g(0) = f(x(0), y(0)) = f(a, b).$$

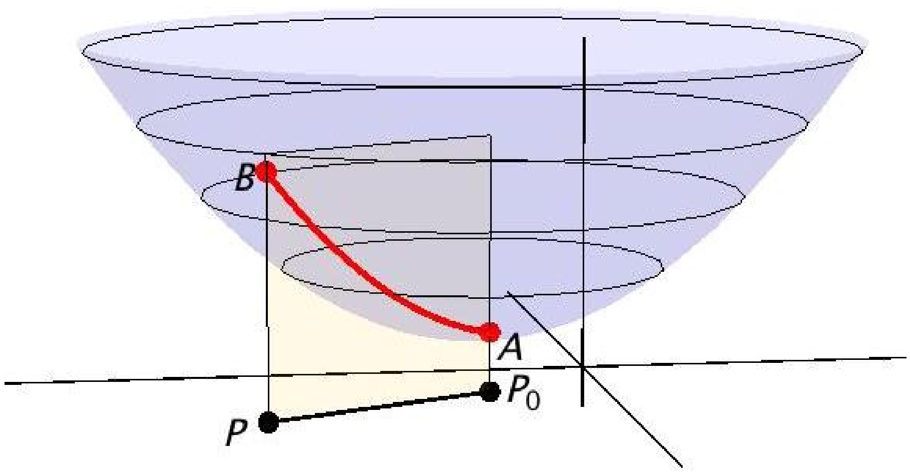
If we assume that $h \in \mathbb{R}$ with $h \neq 0$, we can get that output value of another point on the curve C by evaluating

$$g(h) = f(x(h), y(h)) = f\big(a + hu_1, b + hu_2\big)$$

This yields two points A and B on the curve C with coordinates are given by

$$A\big(a, b, f(a,b)\big) \qquad \text{and} \qquad B\big(a + hu_1, b + hu_2, f(a + hu_1, b + hu_2)\big)$$

As discussed before, we see the two points on the surface in the visual below:



Solution:

III. Measure the slope of the secant line.

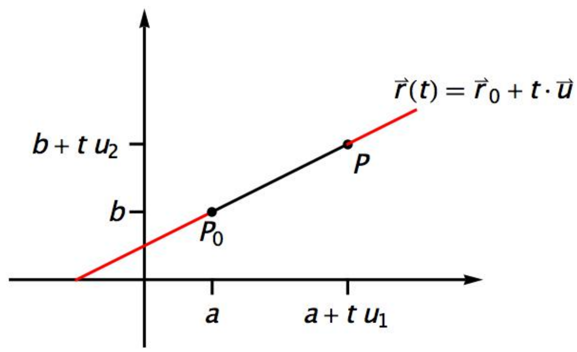
To measure the slope m_{AB} of the secant line through the points A and B, recall that we say that the slope

$$m_{AB} = \frac{\text{change in output}}{\text{signed 'distance' traveled in input}}$$

We can calculate the change in output values on the surface to be given by

$$\text{change in output} = g(h) - g(0) = f\big(a + hu_1, b + hu_2\big) - f(a, b).$$

On the other hand, the signed 'distance' traveled in the input requires some deeper thought. To this end, consider the diagram below:



When moving from point P_0 to point P in the domain, we notice that the scalar h encodes both the magnitude and orientation of this movement. In other words, we see that the nonnegative distance traveled when moving from point P_0 to point P is given by the magnitude:

$$\left\|\overrightarrow{P_0P}\right\|_2 = \|h \cdot \langle u_1, u_2\rangle\|_2 = |h| \cdot \|\mathbf{u}\|_2 = |h|.$$

The fact that the length of this vector is the value of the scalar h directly results from our assumption that $\mathbf{u}$ is a unit vector. To get the signed 'distance' traveled, we remember that in producing the point P, we only required that $h \neq 0$. This corresponds to two scenarios: a positive scalar $h > 0$ or a negative scalar $h > 0$. In each case, the signed 'distance' will just be the value of h. This results in a slope of the secant line through the points A and B given by

$$m_{AB} = \frac{g(h) - g(0)}{h} = \frac{f\big(a + hu_1, b + hu_2\big) - f(a, b)}{h}$$

IV. Transform the secant line into a tangent line using a limit.

V. Construct the "derivative" as the slope of a tangent line.

We recall that we can force point B toward point A by forcing point P to point P_0 in the domain. In particular, we can measure the slope of the tangent line between these points as the following limit:

$$D_{\mathbf{u}}f(a, b) = \lim_{h \to 0} \frac{f\big(a + hu_1, b + hu_2\big) - f(a, b)}{h}$$

2. (4 points) Using the limit definition for the directional derivative of f in the direction of $\mathbf{u}$ at the point (a, b) that you derived in problem 1 above, show how to construct a composite function $g(t)$. This single variable function should have the property that the derivative $g'(t)$ is the same value as the limit we constructed to compute the directional derivative in problem 1.

3. (4 points) Derive the dot product formula for the directional derivative. Be sure to specifically refer to the the function $g(t)$ from problem 2 above along with the multivariable chain rule with two intermediate variables and one independent variables. When appropriate, please explicitly state and use the multivariable chain rule in your work. Also, make sure to explain the value of t that you use to take the ordinary derivative in this derivation.

Solution: By construction, we see that the limit definition of the directional derivative in part A above is given as

$$D_{\mathbf{u}} f(a,b) = \lim_{h \to 0} \frac{f\big(a + hu_1, b + hu_2\big) - f(a,b)}{h}$$

$$= \lim_{h \to 0} \frac{g(0+h) - g(0)}{h}$$

Using ordinary derivative notation, we see this is equivalent to taking the ordinary derivative of the single-variable function

$$g'(0) = \frac{d}{dt}\left[g(t)\right]\bigg|_{t=0}$$

Using the multivariable chain rule, we know

$$\begin{aligned}
\frac{d}{dt}\Big[g(t)\Big]\bigg|_{t=0} &= \frac{d}{dt}\Big[f\big(x(t), y(t)\big)\Big]\bigg|_{t=0} \\
&= \left[\frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}\right]\bigg|_{t=0} \\
&= f_x(a,b) \cdot x'(0) + f_y(a,b) \cdot y'(0) \\
&= f_x(a,b) \cdot u_1 + f_y(a,b) \cdot u_2 \\
&= \langle f_x(a,b), f_y(a,b) \rangle \cdot \langle u_1, u_2 \rangle \\
&= \nabla f(a,b) \cdot \mathbf{u}
\end{aligned}$$

This gives us an alternative method to calculate the directional derivative without requiring limits.

4. (6 points) Using your work in problem 3, explain which unit vectors $\mathbf{u} = \langle u_1, u_2 \rangle$ in the domain D give

   A. the direction of steepest ascent on the surface.
   B. the direction of no change on the surface.
   C. the direction of steepest descent on the surface.

   Please provide evidence that your concept images associated with these directions incorporate multiple categories of knowledge including verbal, graphical, and symbolic representations of these ideas. To earn top scores, your solution should combine the work you did in problem 3 with the cosine formula for the dot product. Also, please make specific connections to between your explanations of each direction and your knowledge of the extreme values of the cosine function.

Solution: By combining our solution in problem 3 above with the cosine formula for the dot product we see

$$\begin{aligned} D_{\mathbf{u}}f(a,b) &= \nabla f(a,b) \cdot \mathbf{u} \\ &= \|\nabla f(a,b)\|_2 \cdot \|\mathbf{u}\|_2 \cdot \cos(\theta) \\ &= \|\nabla f(a,b)\|_2 \cdot \cos(\theta) \end{aligned}$$

where θ is the angle between the vectors $\nabla f(a,b)$ and $\mathbf{u}$. We have three cases to consider.

Case 1: $\theta = 0 \implies \cos(\theta) = 1$

We know that the maximum value of the cosine curve is 1 and this occurs when $\theta = 0$. Applying this knowledge to the directional derivative formula, we know that the derivative $D_{\mathbf{u}}f(a,b)$ has maximum value when $\theta = 0$. Since the directional derivative measured the slope of a tangent line to the surface in the direction of the vector $\mathbf{u}$, the rise over run is a function of which unit vector we choose. The dot product version of the directional derivative indicates that if we want to ascend our surface as quickly as possible, we will get the largest rise over run when $\mathbf{u}$ is the unit vector in the same direction and orientation as the gradient vector $\nabla F(a,b)$. In other words, the slope of this tangent line is maximum when we move in the direction of the gradient vector.

Case 2: $\theta = \dfrac{\pi}{2} \implies \cos(\theta) = 0$

We know that the cosine curve has a zero output value when $\theta = \frac{\pi}{2}$. Applying this knowledge to the directional derivative formula, we know that the derivative $D_{\mathbf{u}}f(a,b)$ is zero when $\theta = \frac{\pi}{2}$. Since the directional derivative measured the slope of a tangent line to the surface in the direction of the vector $\mathbf{u}$, the rise over run is zero in this case. In other words, if we travel 90° from the gradient, we will get no upward or downward motion on the surface. This is equivalent to moving along the contour curve on the surface. Indeed, this unit vector is in the same direction as the tangent line to the level curve of the surface at this point of tangency.

Case 3: $\theta = \pi \implies \cos(\theta) = -1$

We know that the maximum value of the cosine curve is -1 and this occurs when $\theta = \pi$. Applying this knowledge to the directional derivative formula, we know that the derivative $D_{\mathbf{u}}f(a,b)$ has maximum value when $\theta = \pi$. Since the directional derivative measured the slope of a tangent line to the surface in the direction of the vector $\mathbf{u}$, the rise over run is smallest when we travel in the same direction but opposite orientation as the the gradient vector $\nabla F(a,b)$. In other words, we can descend our surface fastest in the negative direction of the gradient vector.

For problems 5 - 6, let $f(x, y) = 15 - x^2 - 4y^2 + 2x - 40y$.

5. (8 points) Find a vector-valued equation for the tangent line to the level curve

$$L_{100}(f) = \{(x, y) \,:\, f(x, y) = 100\}$$

at the point $(-3, -5)$.

Solution: If $D = \mathrm{Dom}(f)$, then we notice that the level curve $L_5(f) \subseteq D \subseteq \mathbb{R}^2$. We begin our work by considering the geometry of this level curve. We notice

$$\begin{aligned} x^2 + y^2 - 6x + 2y - 10 = 5 \quad &\Longrightarrow \quad x^2 - 6x + y^2 + 2y = 15 \\ &\Longrightarrow \quad x^2 - 6x + 9 + y^2 + 2y + 1 = 25 \\ &\Longrightarrow \quad (x-3)^2 + (y+1)^2 = 5^2 \end{aligned}$$

This is a circle with radius $r = 5$ and center point $(h, k) = (3, -1)$.We notice that the given point is on the edge of the circle. To find the vector-valued equation of the tangent line to $L_5(f)$ given by

$$\mathbf{r}(t) = \mathbf{r}_0 + t \cdot \mathbf{v}$$

where $\mathbf{r}_0 \in \mathbb{R}^2$ is a point on the line and $\mathbf{v} \in \mathbb{R}^2$ represents the direction of the line. By the problem statement, we know that $\mathbf{r}_0 = \langle 6, -5 \rangle$. To find the "direction" of this line, we will use implicit differentiation:

$$\begin{aligned} \frac{d}{dy}\Big[x^2 + y^2 - 6x + 2y - 10\Big] = \frac{d}{dy}\Big[5\Big] \quad &\Longrightarrow \quad 2x - 6 + 2y \cdot \frac{dy}{dx} + 2 \cdot \frac{dy}{dx} = 0 \\ &\Longrightarrow \quad \frac{dx}{dy} = \frac{3-x}{y+1} \\ &\Longrightarrow \quad \left.\frac{dx}{dy}\right|_{(6,-5)} = \frac{3}{4} \\ &\Longrightarrow \quad \mathbf{v} = \langle 4, 3 \rangle \end{aligned}$$

Using this calculation, we find

$$\mathbf{r}(t) = \langle 6, -5 \rangle + t \cdot \langle 4, 3 \rangle = \langle 6 + 4t \,,\, -5 + 3t \rangle$$

6. (6 points) On the axes below, sketch the level curve $L_{100}(f)$ and it's the tangent line from problem 5 above. Also, sketch the vector $\mathbf{u} \in \mathbb{R}^2$ with tail at point $(-3, -5)$ where $\mathbf{u}$ is the unit vector in the direction of the gradient vector $\nabla f(-3, -5)$ given by

$$\mathbf{u} = \frac{\nabla f(-3, -5)}{\|\nabla f(-3, -5)\|_2}$$

Solution: We begin this problem by finding the gradient of our function at the given point:

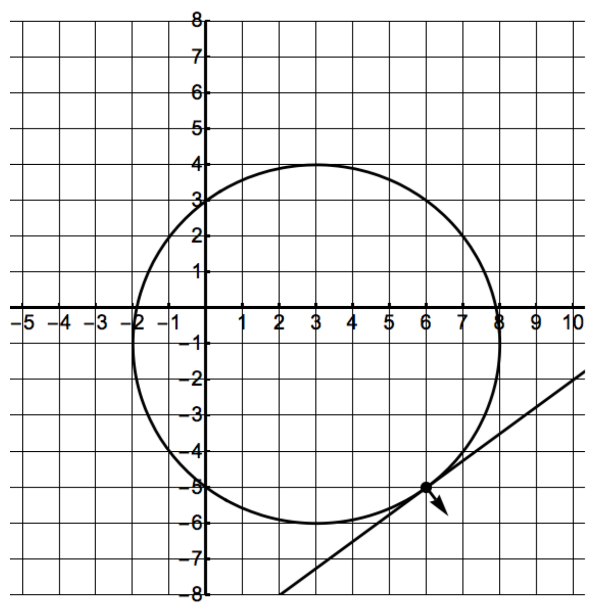
$$\nabla f(6, -5) = \langle\, 2x - 6\,,\, 2y + 2\,\rangle\,\big|_{(6,-5)}$$

$$= \langle\, 6\,,\, -8\,\rangle$$

Then, we can see that

$$\mathbf{u} = \frac{\nabla f(6, -5)}{\|\nabla f(6, -5)\|_2} = \left\langle\, \frac{3}{5}\,,\, -\frac{4}{5}\,\right\rangle$$

We graph this vector with tail $(6, -5)$ below.



Now, use full sentences to explain how your graph above relates your knowledge about the shape of the surface $f(x, y)$ and your solution to problem 6 above.

Solution: Notice that the surface is an upward facing elliptic parabola. The vertex of this surface is at the point $(3, -1, -20)$. Based on the shape of the surface, we know that at the input point $(6, -5)$, the direction of fastest ascent is directly away the center point of each circular level curve. Indeed, this is what we see with our gradient vector at this point.

For problems 7, 8, and 9, choose two out of these three problems you want me to grade for your first attempt during our in-class exam session. For the problem you would like to skip grading for your in-class attempt, please mark a big "X" through that problem. For the problem you skip, you can submit your solutions in your exam corrections. For now, focus on the two of these problems that you feel most comfortable with and give your best effort.

7. (8 points) Among all the points on the graph of $z = 10 - x^2 - y^2$ that lie above the plane $x + 2y + 3z = 0$, find the point farthest from the plane.

Solution: Let's define the distance from the plane to the point using the function:

$$\begin{aligned} d(x, y, z) &= \sqrt{(x-2)^2 + (y-0)^2 + (z - -3)^2} \\ &= \sqrt{(x-2)^2 + y^2 + (z+3)^2} \end{aligned}$$

By the equation for the plane, we know that $z = 1 - x - y$. Thus we can write the distance function in terms of x and y:

$$\begin{aligned} d(x, y) &= \sqrt{(x-2)^2 + y^2 + (1 - x - y + 3)^2} \\ &= \sqrt{(x-2)^2 + y^2 + (4 - x - y)^2} \end{aligned}$$

Notice that function $d(x, y)$ achieve a minimum at point (a, b) if and only if function $d^2(x, y)$ has a minimum value at point (a, b). Thus, to find the minimum distance, let's study the function

$$f(x, y) = d^2(x, y) = (x-2)^2 + y^2 + (4 - x - y)^2.$$

By the 2nd derivative test for multivariable functions, we know that the minimum of d^2 will occur where $\nabla f(x, y) = \mathbf{0}$:

$$\nabla f(x, y) = \begin{bmatrix} 2(x-2) - 2(4 - x - y) \\ 2y - 2(4 - x - y) \end{bmatrix} = \begin{bmatrix} 4x + 2y - 12 \\ 2x + 4y - 8 \end{bmatrix}$$

Thus we see that $\nabla f(x, y) = \mathbf{0}$ if and only if $x = 8/3$ and $y = 2/3$. We can further check the Wronskian of $f(x, y)$ given by

$$f_{xx} \cdot f_{yy} - f_{xy}^2 = 4 \cdot 4 - 2^2 = 12 > 0.$$

Since the Wronskian is positive and $f_{xx} = 4$, we know that f(x,y) has a local minimum at point $(x, y) = (0.5, 2)$. Thus, the minimum distance will occur at the point:

$$\boxed{\left(\frac{8}{3}, \frac{2}{3}, -\frac{7}{3}\right)}$$

8. (8 points) Consider the function

$$f(x, y) = (x-1)^2 + (y-2)^2$$

Find the minimum and maximum values of $f(x, y)$ subject to the constraint that $x^2 + y^2 = 45$.

Solution: Let's define the constraint function

$$g(x, y) = x^2 + y^2 - 1$$

We will solve this problem using Lagrange multipliers. To this end, we want to find values of (x, y) and λ such that

$$\text{Eq. 1}: \qquad \frac{2x}{49} = 2\lambda x$$

$$\text{Eq. 2}: \qquad \frac{2y}{4} = 2\lambda y$$

$$\text{Eq. 3}: \qquad 0 = x^2 + y^2 - 1$$

We can use the zero-product property on equations 1 and 2 to find two possible solutions for each equations:

$$2x \cdot \left(\frac{1}{49} - \lambda\right) = 0 \qquad \Longrightarrow \qquad x = 0 \text{ or } \lambda = \frac{1}{49}$$

$$2y \cdot \left(\frac{1}{4} - \lambda\right) = 0 \qquad \Longrightarrow \qquad y = 0 \text{ or } \lambda = \frac{1}{4}$$

This results in four points

$$(0, \pm 1) \qquad \text{or} \qquad (\pm 1, 0)$$

This correlates to the minimum value of $\dfrac{1}{49}$ and maximum value $\dfrac{1}{4}$.

9. (8 points) Consider two multivariable functions defined by implicit equations

$$F(x, y, z) = x^2 + y^2 - 2 = 0 \qquad \text{and} \qquad G(x, y, z) = x + z - 4 = 0.$$

Notice that $F(x, y, z) = 0$ defines a cylinder while $G(x, y, z) = 0$ defines a plane. The intersection of these two surfaces forms an ellipse E. Find the parametric equation for the line tangent to E at the point $P_0(1, 1, 3)$.

Solution: Let's define the constraint function

$$g(x, y) = x^2 + y^2 - 1$$

We will solve this problem using Lagrange multipliers. To this end, we want to find values of (x, y) and λ such that

$$\text{Eq. 1}: \qquad \frac{2x}{49} = 2\lambda x$$

$$\text{Eq. 2}: \qquad \frac{2y}{4} = 2\lambda y$$

$$\text{Eq. 3}: \qquad 0 = x^2 + y^2 - 1$$

We can use the zero-product property on equations 1 and 2 to find two possible solutions for each equations:

$$2x \cdot \left(\frac{1}{49} - \lambda\right) = 0 \qquad \Longrightarrow \qquad x = 0 \text{ or } \lambda = \frac{1}{49}$$

$$2y \cdot \left(\frac{1}{4} - \lambda\right) = 0 \qquad \Longrightarrow \qquad y = 0 \text{ or } \lambda = \frac{1}{4}$$

This results in four points

$$(0, \pm 1) \qquad \text{or} \qquad (\pm 1, 0)$$

This correlates to the minimum value of $\frac{1}{49}$ and maximum value $\frac{1}{4}$.