Lesson 10: The Chain Rule (for Multivariable functions)

Recall the chain Rule for single variable functions

$$\frac{d}{dx}\left[f(g(x))\right] = \underbrace{f'(g(x))}_{\substack{\text{The derivative of} \\ \text{the outer function} \\ \text{evaluated at the} \\ \text{inner function}}} \cdot \underbrace{g'(x)}_{\substack{\text{the derivative} \\ \text{of the inner function}}}$$

Proof: (special case) Let $f(x)$ be differentiable at input value $u = g(a)$ and let $g(x)$ be differentiable at a.

Set $c(x) = f(g(x)) = f \circ g(x)$.

By the limit definition of derivative (in slope notation), we have

$$c'(a) = \lim_{x \to a} \frac{c(x) - c(a)}{x - a}$$

$$= \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a}$$

Assume that $g(a) \neq g(x)$ for all values of x near a with $x \neq a$. Then, if we set $v = g(x)$ and $u = g(a)$, we can write

$$c'(a) = \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x-a}$$

$$= \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x-a}$$

$$= \lim_{x \to a} \frac{f(v) - f(u)}{v-u} \cdot \frac{g(x) - g(a)}{x-a}$$

Since $g(x)$ is differentiable at a, we know $g(x)$ is continuous at a so $\lim_{x \to a} g(x) = g(a)$ and $v \to u$ as $x \to a$. Thus

$$c'(a) = \lim_{v \to u} \frac{f(v) - f(u)}{v-a} \cdot \lim_{x \to a} \frac{g(x) - g(a)}{x-a} = f'(g(a)) \cdot g'(a). \quad \square$$

We will now generalize the idea of the chain Rule to multivariable functions. We will do this generalization using a few different scenarios.

The Chain Rule: Case 1

Suppose $z = f(x,y)$ is a differentiable function of x and y, where

$$x = g(t)$$
$$y = h(t)$$

are both differentiable functions of t.

Then, z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

Proof:

similar to
Example 12.5.1 p. 909

Let $z = x^2 y + 3xy^4$ where $x(t) = \sin(2t)$, $y(t) = \cos(t)$

Let's find $\frac{dz}{dt}$ when $t = 0$.

Note: at $t = 0$

$x(0) = \sin(0) = 0$
$y(0) = \cos(0) = 1$ } $(0, 1)$

Solution: Recall from the chain rule (case 1)

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$= \left[2xy + 3y^4\right] \cdot \left(2\cos(2t)\right) + \left[x^2 + 12xy^3\right]\left(-\sin(t)\right)$$

Now, the derivative of z w/ respect to t is given by

$$\left.\frac{dz}{dt}\right|_{\cancel{(0,1)}\, t=0} = \left[2 \cdot 0 \cdot 1 + 3 \cdot 1^4\right] \cdot \left(2\cos(0)\right) + \left[0^2 + 12 \cdot 0 \cdot 1^2\right]\left(-\sin(0)\right)$$

$$= 3 \cdot 2$$

$$= \boxed{6}.$$

The Chain Rule: Case 2

Suppose $z = z(x, y)$ is a differentiable function of x and y,

where $x = x(s,t)$ and $y = y(s,t)$ are

differentiable functions of s and t.

Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$z = f(x, y)$, $x = g(s,t)$, $y = h(s,t)$

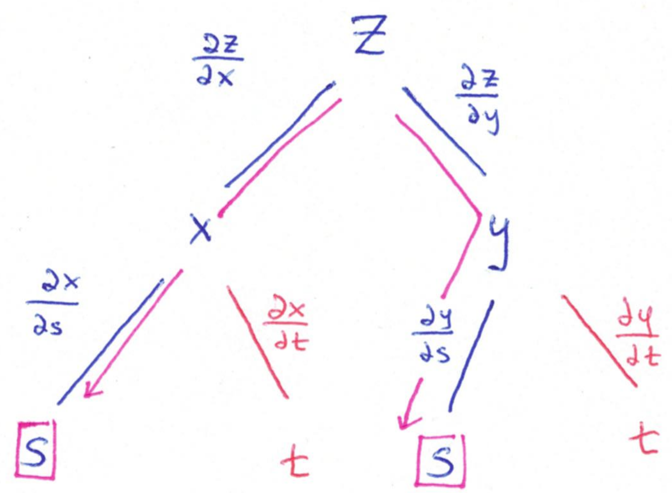
z - dependendent variables

x, y - intermediate variables

s, t - independent variables

We can use tree diagrams to ~~calcul~~ help us remember how to calculate derivatives using multivariable chain rule :

$$z = f(x, y), \quad x = g(s,t), \quad y = h(s,t)$$



$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$ follow the tree

Similar to
Example 12.5.2 p. 910

Let $z = e^{x} \sin(y)$ where $x = s \cdot t^2$ & $y = s^2 t$

Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$

Solution: Using Case 2 of Multivariable chain Rule, we see

$$\frac{\partial z}{\partial s} = \left[\frac{\partial z}{\partial x}\right] \cdot \left(\frac{\partial x}{\partial s}\right) + \left[\frac{\partial z}{\partial y}\right]\left(\frac{\partial y}{\partial s}\right)$$

$$= \left[e^{x} \sin(y)\right]\left(t^2\right) + \left[e^{x} \cos(y)\right]\left(2st\right)$$

$$= t^2 e^{st^2} \sin(s^2 t) + 2st\, e^{st^2} \cos(s^2 t) \ \checkmark$$

$$\frac{\partial z}{\partial t} = \left[\frac{\partial z}{\partial x}\right] \cdot \left(\frac{\partial x}{\partial t}\right) + \left[\frac{\partial z}{\partial y}\right]\left(\frac{\partial y}{\partial t}\right)$$

$$= \left[e^{x} \sin(y)\right] \cdot (2st) + \left[e^{x} \cos(y)\right] \cdot \left(s^2\right)$$

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$$= 2st\, e^{st^2} \sin(s^2 t) + s^2 e^{st^2} \cos(s^2 t) \ \checkmark$$

The Chain Rule: General Version

Suppose $u = f(x_1, x_2, \ldots, x_n)$ and (w/ f differentiable)

suppose $x_i = g_i(t_1, t_2, \ldots, t_m)$ differentiable

Then u is a differentiable function of t_i

and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_i}$$

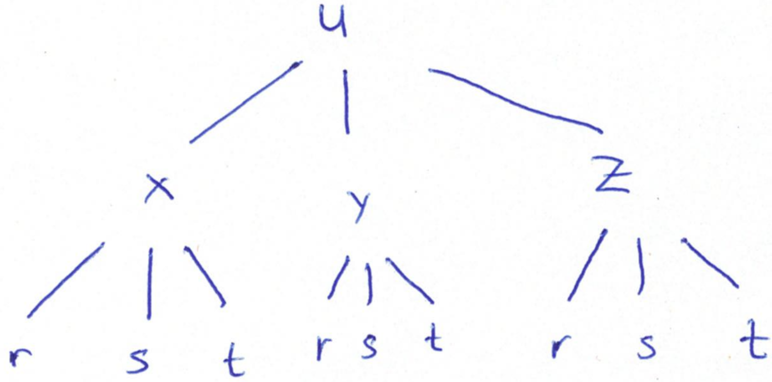
$$= \sum_{j=1}^{n} \frac{\partial u}{\partial x_j} \cdot \frac{\partial x_j}{\partial t_i}$$

Similar to

Example 12.5.3 p. 910

Let $u = x^4 y + y^2 z^3$ where

$x = r s e^t, \quad y = r s^2 e^{-t}, \quad z = r^2 s \sin(t)$



Find $\dfrac{\partial u}{\partial s}$ when $r = 2, \; s = 1, \; t = 0.$

Solution: Using the tree diagram and our general chain rule, we see that

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial s}$$

Let's evaluate each of these separately. We begin by finding the values of x, y & z at our stated point.

If $r=2$, $s=1$ and $t=0$, we have

$$x = r \cdot s \cdot e^{t} = 2 \cdot 1 \cdot e^{0} = 2$$

$$y = r \cdot s^{2} e^{-t} = 2 \cdot 1^{2} \cdot e^{0} = 2$$

$$z = r^{2} \cdot s \cdot \sin(t) = 2^{2} \cdot 1 \cdot \sin(0) = 0$$

Then, using this data, we can find

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[x^{4} \cdot y + y^{2} z^{3}\right]$$

$$= 4x^{3} \cdot y$$

$$\Rightarrow \left.\frac{\partial y}{\partial x}\right|_{(2,2,0)} = 4 \cdot 2^{3} \cdot 2 = 2^{2} \cdot 2^{3} \cdot 2^{1} = 2^{6} = \boxed{64}$$

On the other hand

$$\frac{\partial x}{\partial s} = \frac{\partial}{\partial s}\left[r s e^{t}\right] = r \cdot e^{t}$$

$$\Rightarrow \left.\frac{\partial x}{\partial s}\right|_{(2,1,0)} = 2 \cdot e^{0} = \boxed{2}$$

Next, we see

$$\left.\frac{\partial u}{\partial y}\right|_{(x,y,z)=(2,2,0)} = \left.\frac{\partial}{\partial y}\left[x^4 \cdot y + y^2 \cdot z^3\right]\right|_{(2,2,0)}$$

$$= \left. x^4 + 2yz^3 \right|_{(2,2,0)}$$

$$= 2^4 + 2 \cdot 2 \cdot 0^3$$

$$= \boxed{16}$$

We also know

$$\left.\frac{\partial y}{\partial s}\right|_{(r,s,t)=(2,1,0)} = \left.\frac{\partial}{\partial s}\left[r \cdot s^2 \cdot e^{-t}\right]\right|_{(2,1,0)}$$

$$= \left. 2 \cdot r \cdot s \cdot e^{-t} \right|_{(2,1,0)}$$

$$= 2 \cdot 2 \cdot 1 \cdot e^0$$

$$= \boxed{4}$$

Finally, we have

$$\left.\frac{\partial u}{\partial z}\right|_{(x,y,z)=(2,2,0)} = \left.\frac{\partial}{\partial z}\left[x^4 \cdot y + y^2 z^3\right]\right|_{(2,2,0)}$$

$$= \left. 3y^2 z^2 \right|_{(2,2,0)}$$

$$= 3 \cdot 2^2 \cdot 0$$

$$= 0$$

For completeness, we find

$$\left.\frac{\partial z}{\partial s}\right|_{(r,s,t)=(2,1,0)} = \left.\frac{\partial}{\partial s}\left[r^2 \cdot s \cdot \sin(t)\right]\right|_{(2,1,0)}$$

$$= \left. r^2 \cdot \sin(t) \right|_{(2,1,0)}$$

$$= 2^2 \cdot \sin(0)$$

$$= 0 .$$

Thus, $\left.\frac{\partial u}{\partial s}\right|_{(2,1,0)} = 64 \cdot 2 + 16 \cdot 4 + 0 \cdot 0 = \boxed{192}$