Lesson 13: Maximum and Minimum Problems Handout
Reference: Brigg's "Calculus: Early Transcendentals, Second Edition"
Topics: Section 12.5: Maximum and Minimum Problems, p. 939 - 951

12.8 Maximum and Minimum Problems p. 939 - 951

Definition. ***Local Maximum Value(s)*** *p. 939*

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a two variable function. We say that f has a local maximum at (a, b) if and only if

$$f(x, y) \leq f(a, b)$$

for (x, y) in the domain of f in some open disk centered at (a, b). We call this output value $f(a, b)$ the **local maximum value** on this open disk $D \subseteq \mathbb{R}^2$ since

$$f(a, b) = \max_{\mathbf{x} \in D} f(\mathbf{x})$$

In this case, we call the input point (a, b) a **local maximizer** of the function f since

$$(a, b) = \arg\max_{\mathbf{x} \in D} f(\mathbf{x})$$

Definition. ***Local Minimum Value(s)*** *p. 939*

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a two variable function. We say that f has a local minimum at (a, b) if and only if

$$f(x, y) \geq f(a, b)$$

for (x, y) in the domain of f in some open disk centered at (a, b). We call this output value $f(a, b)$ the **local minimum value** on this open disk $D \subseteq \mathbb{R}^2$ since

$$f(a, b) = \min_{\mathbf{x} \in D} f(\mathbf{x})$$

In this case, we call the input point (a, b) a **local minimizer** of the function f since

$$(a, b) = \arg\min_{\mathbf{x} \in D} f(\mathbf{x})$$

Theorem 12.13. ***Necessary conditions for unconstrained optimization problems*** *p. 939*

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a two variable function. If f has a local maximum or local minimum at (a, b) and if $f(x, y)$ is differentiable at the point (a, b), then $\nabla f(a, b) = 0$ (i.e. $f_x(a, b) = f_y(a, b) = 0$).

Definition. ***Critical Point*** *p. 940*

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a two variable function. An interior point (a, b) in the domain of the function f is a **critical point** of f if and only if either of the following is true:

1. $\nabla f(a, b) = 0$ (i.e. the first partials $f_x(a, b) = f_y(a, b) = 0$)
2. at least one of the partial derivatives f_x or f_y does not exist at the point (a, b)

Theorem 12.14. ***Second Partial Derivatives Test*** *p. 941*

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a two variable function. Suppose that $f(x, y)$ is twice differentiable on an open disk centered at the point (a, b) where $\nabla f(a, b) = 0$. Define the **discriminant** of f to be the function

$$D(x, y) = f_{xx}(x, y) \cdot f_{yy}(x, y) - (f_{xy}(x, y))^2 \tag{12.1}$$

Then, we can use this function to make the following conclusions:

1. If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then f has a local maximum value at (a, b)

2. If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then f has a local minimum value at (a, b)

3. If $D(a, b) < 0$, then f has a saddle point at (a, b)

4. If $D(a, b) = 0$, then this test is inconclusive and cannot be used to identify the behavior of f at point (a, b)

Definition. ***Saddle Point*** *p. 940*

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a two variable function. The function f is said to have a **saddle point** at the critical point (a, b) if and only if in every disk centered at (a, b):

1. there is at least one point (x, y) at which $f(x, y) > f(a, b)$
2. here is at least one (different) point (x, y) at which $f(x, y) < f(a, b)$

Procedure. ***Finding Absolute Maximum and Minimum on a Closed Set*** *p. 944*

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function on a closed and bounded set $R \subseteq \mathbb{R}^2$. To find the absolute maximum and minimum values of f on R:

1. Find the output values of f at all critical points in R.

2. Find the maximum and minimum values of f on the boundary of R.

3. Remember that the greatest output value(s) of f found in Steps 1 and 2 is the absolute maximum of f on R.

4. Remember that the least output value(s) of f found in Steps 1 and 2 is the absolute minimum of f on R.