Math 1C: Calculus III

Lesson 14: Lagrange Multipliers

Reference: Brigg's "Calculus: Early Transcendentals, Second Edition"
Topics: Section 12.8: Maximum and Minimum Problems, p. 951 - 959

Definition. ***Two-Variable Constrained Optimization Problem*** *p. 951*

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a two variable function. In constrained optimization problem, we seek to find the maximum or minimum value of the **objective function** $f(x, y)$ subject to the restriction that the values of x and y must lie on a **constraint** curve C in the xy−plane given by $g(x, y) = 0$. A general formulation of a constrained maximization problem is written

$$\max_{(x,y)\in\mathbb{R}^2} f(x, y) \qquad \text{subject to } g(x, y) = 0.$$

On the other hand, a constrained minimization problem can be written as

$$\min_{(x,y)\in\mathbb{R}^2} f(x, y) \qquad \text{subject to } g(x, y) = 0.$$

Example 12.8.7 p. 946) Solve the following constrained optimization problem

$$\max_{(x,y)\in\mathbb{R}^2} \underbrace{x^2 + y^2 - 2x + 2y + 5}_{\text{objective function}}$$

on the curve $C: \{(x,y) : \underbrace{x^2 + y^2 = 4}_{\text{constraint}}\}$

(careful: not $g(x,y)$)

Example 12.8.7 p. 946...)

Solution: The first thing we do is to transform our problem into general form. To this end, let's define objective function

$$f(x,y) = x^2 - 2x + y^2 + 2y + 5$$

$$= (x-1)^2 + (y+1)^2 + 3$$

we notice this is a paraboloid with vertex at point $(1,-1)$ (which is the global minimum of $f(x,y)$ over all $(x,y) \in \mathbb{R}^2$)

Next, we define the constraint

$$g(x,y) = x^2 + y^2 - 4 = 0$$

that gives rise to curve C.

We further notice that C is a circle of radius 2 centered at $(0,0)$ corresponding to level curve $g(x,y) = 0$.

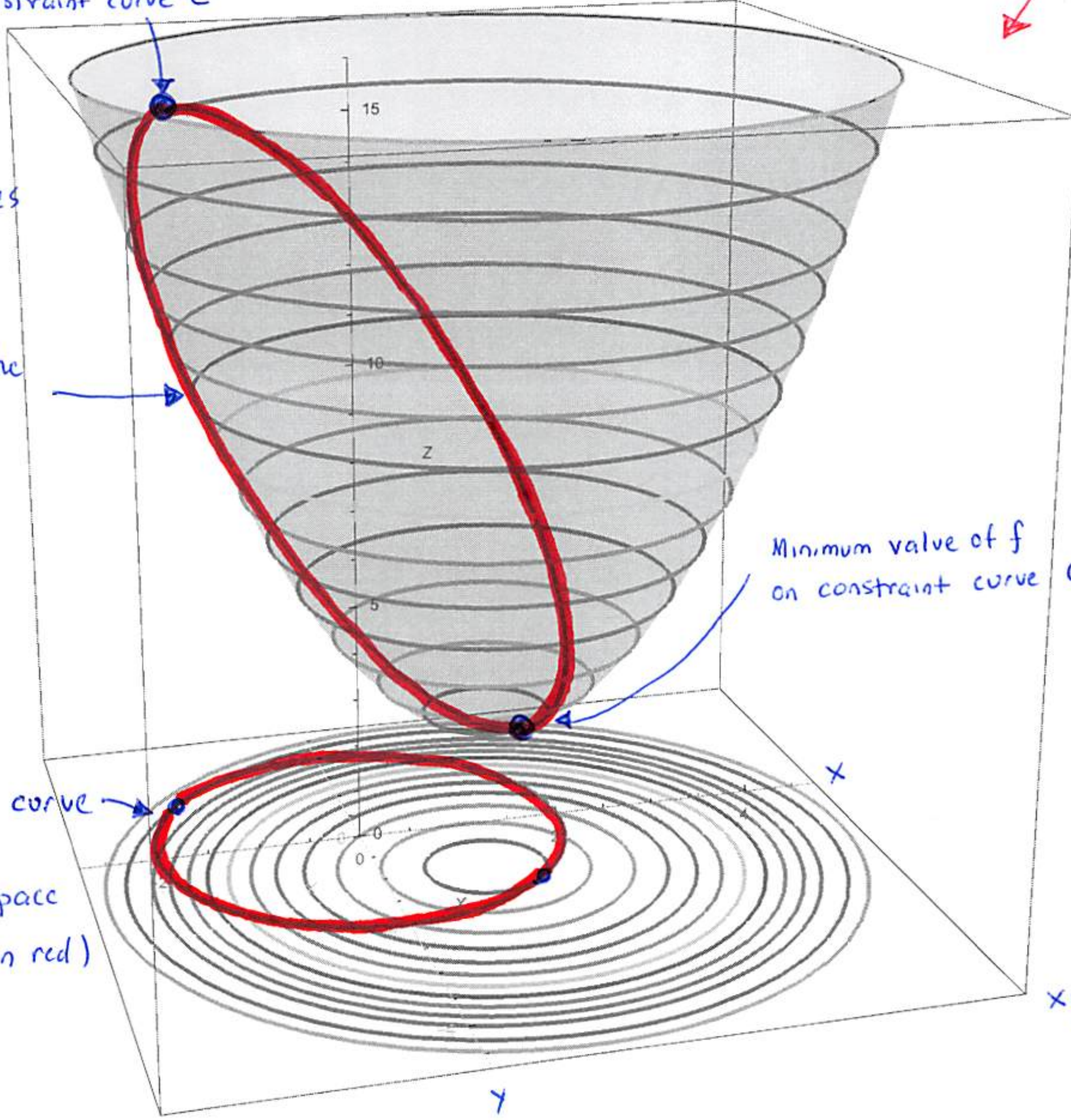
Example 12.8.7 p. 946 ...)

see Mathematica notebook associated with lesson 14 to play with 3D graphic here

Surface Plot:
$f(x,y) = (x-1)^2+(y+1)^2+3$

Maximum value of f on constraint curve C

Output values of f(x,y) for all (x,y) on the constraint curve



Minimum value of f on constraint curve C

Constraint curve C in Domain space (Highlighted in red)

xy-plane

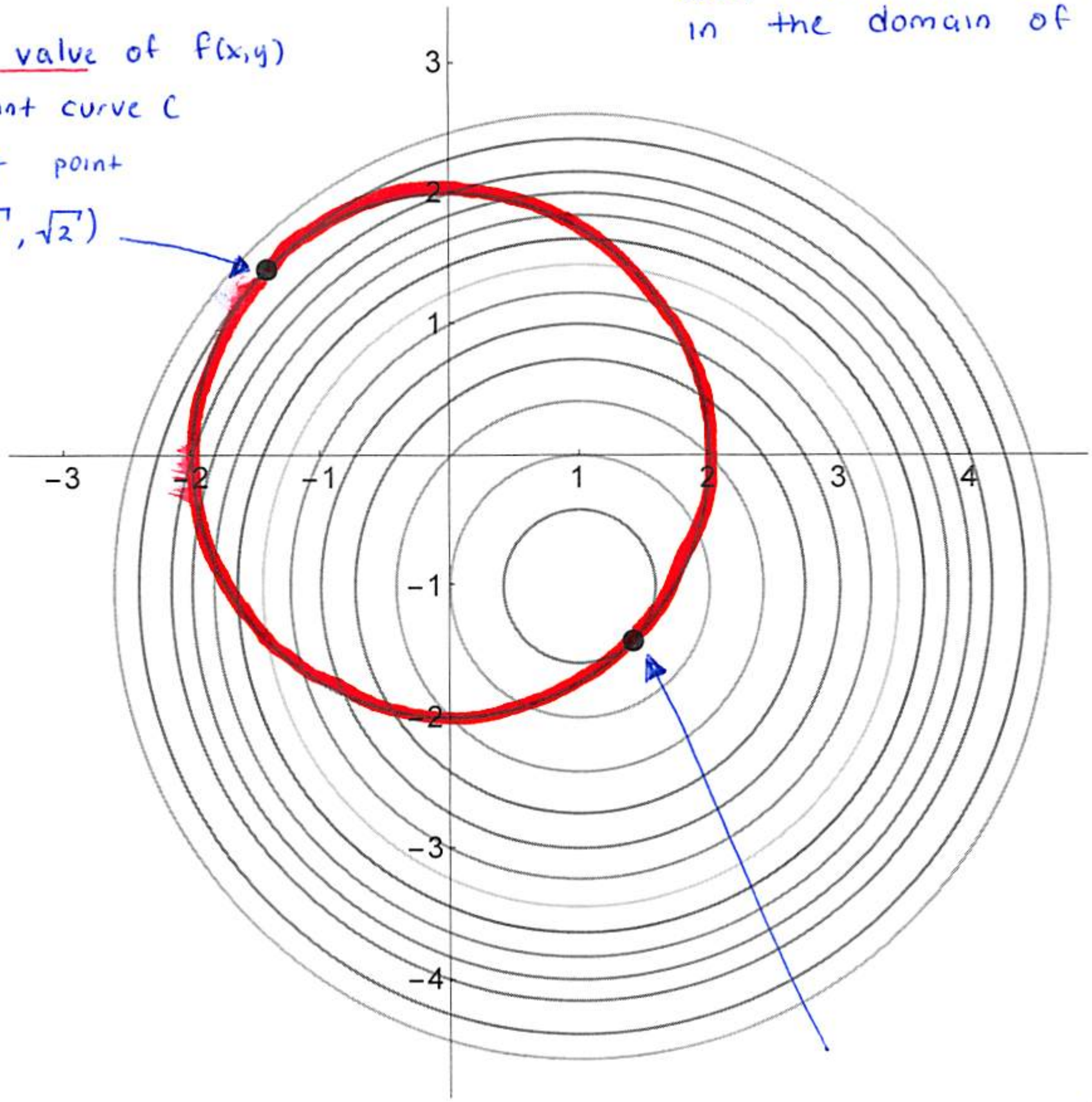
This Figure shows the level curves of f(x,y) in the xy-plane highlighted in various colors.

Moreover, the output values of f(x,y) corresponding to each level curve are highlighted on the surface in the same color as the associated level curve in the xy-plane

Example 12.8.7 p. 946 ...)

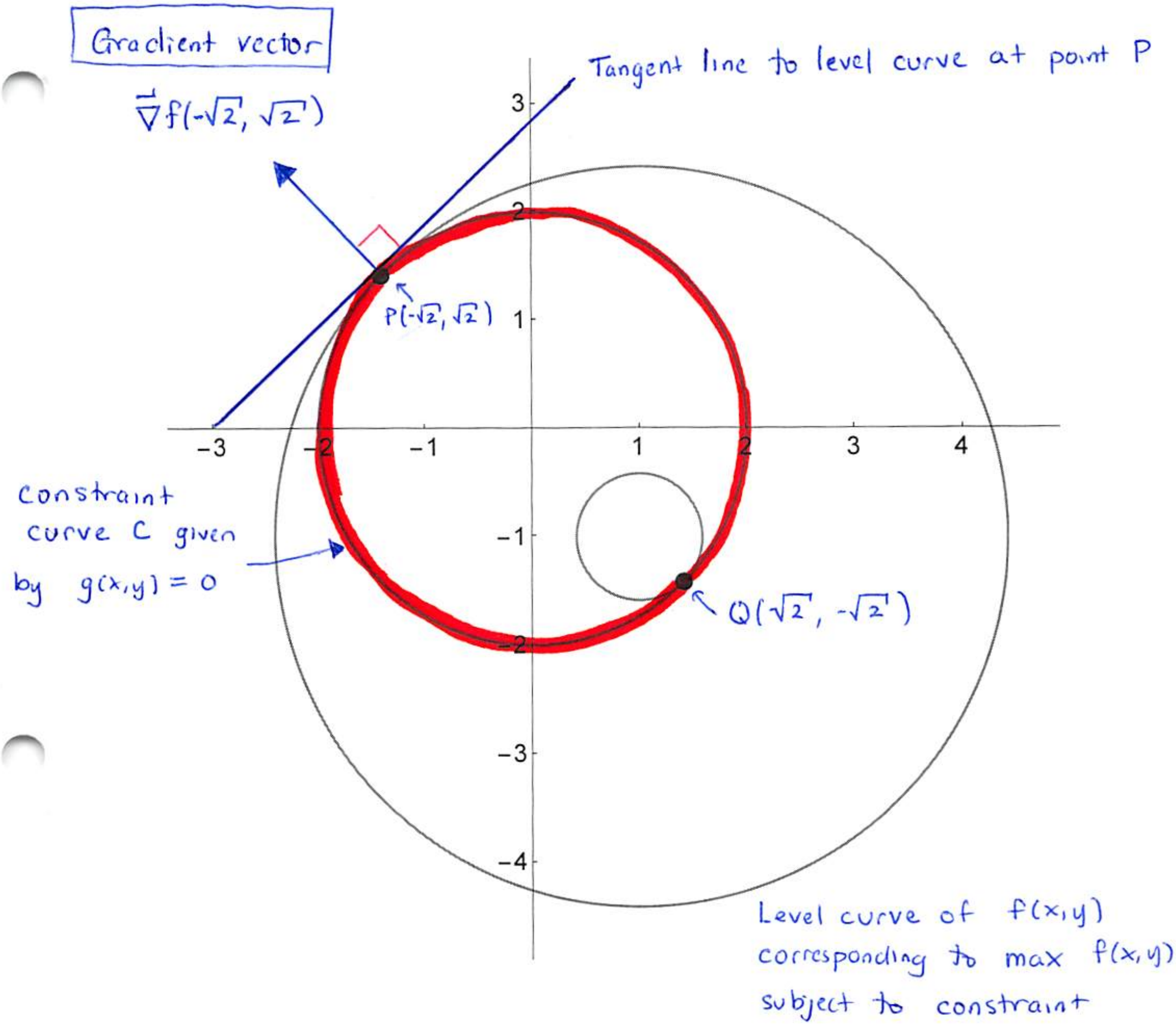
Level curves of $f(x,y)$ and constraint curve C in the domain of $f(x,y)$

Maximum value of $f(x,y)$ on constaint curve C occurs at point $P(-\sqrt{2}, \sqrt{2})$



Minimum value of $f(x,y)$ on constraint curve C occurs at point $Q(\sqrt{2}, -\sqrt{2})$.

There is something very special going on at points P and Q.



Recall, by Theorem 12.12 p. 922 that the gradient of $f(x,y)$, given by $\vec{\nabla} f(x,y)$, is orthogonal to the tangent line of the level curve at (x,y)

However, if we consider the curve C, given by $g(x,y) = 0$, as a level curve of $g(x,y)$, notice $\nabla g(-\sqrt{2}, \sqrt{2})$ is also orthogonal to the same line (i.e. parallel to ∇f).

Example 12.8.7 p. 946 ...)

In other words vectors

$$\vec{\nabla} f(-\sqrt{2}, \sqrt{2}) \qquad \text{and} \qquad \vec{\nabla} g(-\sqrt{2}, \sqrt{2})$$

are parallel. So, we must be able to find

some scalar $\lambda \in \mathbb{R}$ so that

this scalar is called a Lagrange multiplier

$$\vec{\nabla} f = \lambda \cdot \vec{\nabla} g$$

along the curve C given by equation

$$g(x, y) = 0.$$

Theorem 12.15. ***Parallel Gradients (Ball Park Theorem** p. 952*

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a two-variable, differentiable function in a region $D \subseteq \mathbb{R}^2$. Suppose that a smooth curve C, defined by the equation $g(x, y) = 0$, lies in the interior of region D. Assume that $f(x, y)$ has a local extreme value on C at a point $P(a, b)$. Then, $\nabla f(a, b)$ is orthogonal to the line tangent to the curve C at point $P(a, b)$. In other words, assuming that $\nabla g(a, b) \neq \mathbf{0}$, it follows that there should be some real number λ, called a **Lagrange multiplier**, such that

$$\nabla f(a, b) = \lambda \nabla g(a, b).$$

Proof: Let all assumptions given in the theorem statement hold.

We Know that curve C is smooth and thus we can represent C parametrically in the form

$$C: \vec{r}(t) = \langle x(t), y(t) \rangle$$

where $x(t)$ and $y(t)$ are differentiable functions of t on an interval that contains value t_0 with

$$\vec{r}(t_0) = \langle x(t_0), y(t_0) \rangle = P(a,b).$$

As we move along C, we can find the rate of change of $f(\vec{r}(t))$ given by

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

← applying the multivariable Chain rule (one independent variable) Thm 12.7 p. 908

$\boxed{\text{L14, p. 7}}$

$$\Rightarrow \quad \frac{df}{dt} = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$$

$$= \vec{\nabla} f(x,y) \cdot \vec{r}\,'(t)$$

Now, at the point $P(a,b) = (x(t_0), y(t_0))$, the single variable, differentiable function

$$f(\vec{r}(t))$$

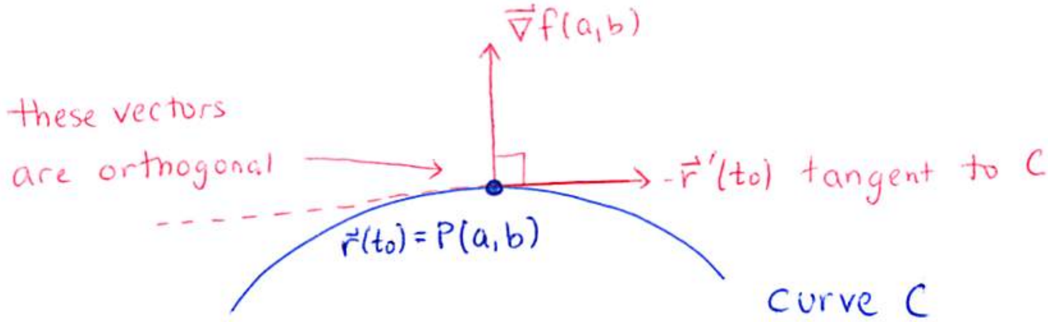
has a local maximum or local minimum. By our Local Extreme Value Theorem 4.2 p. 239, we know

$$\left.\frac{df}{dt}\right|_{t=t_0} = 0$$

$$\Rightarrow \quad \vec{\nabla} f(a,b) \cdot \vec{r}\,'(t_0) = 0$$

$$\Rightarrow \quad \vec{\nabla} f(a,b) \perp \vec{r}\,'(t_0)$$

However, we know that $\vec{r}\,'(t_0)$ is tangent to the curve C at point P (see diagram on next page).



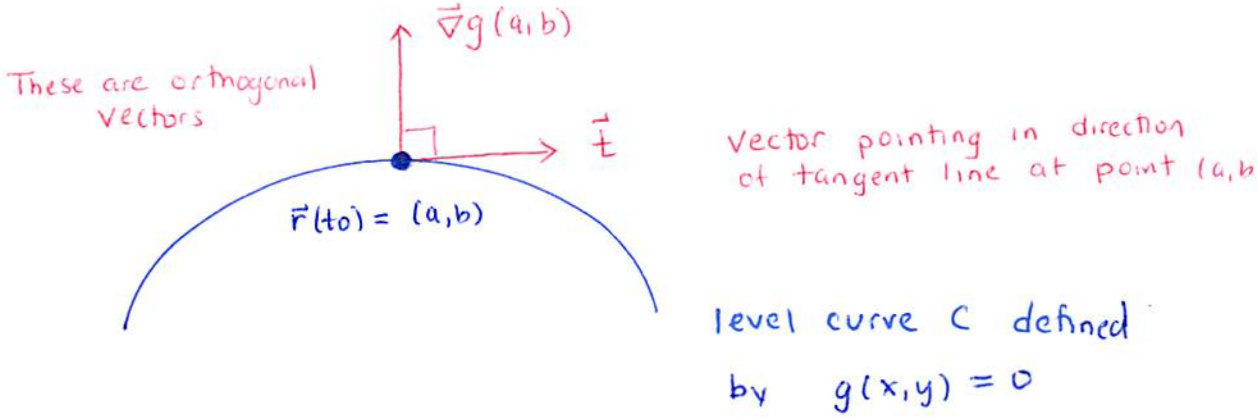
Thus, we see $\vec{\nabla} f(a,b)$ is orthogonal to the line tangent to C at point P. This is the first assertion we wanted to prove.

Next, we want to show $\vec{\nabla} g(a,b)$ is parallel to $\vec{\nabla} f(a,b)$.

To this end, notice that the constraint curve C given by $g(x,y) = 0$ is a level curve of the surface

$$z = g(x,y).$$

Recall by Theorem 12.12 p. 422 that $\vec{\nabla} g(x,y)$ is orthogonal to the tangent line at point (x,y) on level curve C (see diagram on next page)



Thus, at the point $\vec{r}(t_0) = P(a,b)$ on curve C,

we know that $\vec{\nabla} g(a,b)$ is orthogonal to the tangent line to C.

But, $\vec{\nabla} f(a,b)$ and $\vec{\nabla} g(a,b)$ are vectors in $\mathbb{R}^2$ that are both orthogonal to the same line.

Thus, $\vec{\nabla} f(a,b)$ and $\vec{\nabla} g(a,b)$ must be parallel.

Algebraically, these vectors are parallel if and only if we can find a scalar λ such that

$$\vec{\nabla} f(a,b) = \lambda \cdot \vec{\nabla} g(a,b)$$

Procedure. *Method of Lagrange Multipliers in Two Variables p. 953*

Let the objective function $f : \mathbb{R}^2 \to \mathbb{R}$ and the constraint function $g : \mathbb{R}^2 \to \mathbb{R}$ be differentiable on a region in $\mathbb{R}^2$. Assume that $\nabla g(x, y) \neq \mathbf{0}$ on the curve C defined by the equation $g(x, y) = 0$. To locate the maximum or minimum values of $f(x, y)$ subject to the constraint $g(x, y) = 0$, we carry out the following steps:

1. Find all the values of x, y and λ (if they exist) that satisfy the equations:

$$\nabla f(x, y) = \lambda \nabla g(x, y) \qquad \text{and} \qquad g(x, y) = 0.$$

2. Among the values of (x, y) found in Step 1, select the largest and smallest corresponding function values $f(x, y)$. These values are the maximum and minim values of f subject to the given constraint.

Notice that the equation

$$\langle f_x(x,y), f_y(x,y) \rangle = \vec{\nabla} f(x,y) = \lambda \vec{\nabla} g(x,y) = \lambda \langle g_x(x,y), g_y(x,y) \rangle$$

is a vector equation. Thus, the method of Lagrange multipliers in two variables depends on solving three equations in three unknowns

Equation 1: $f_x(x,y) = \lambda \cdot g_x(x,y)$

Equation 2: $f_y(x,y) = \lambda \, g_y(x,y)$

Equation 3: $g(x,y) = 0$

the unknowns in these equations are variables x, y, λ.

Note: Equations 1 & 2 result from $\vec{\nabla} f = \lambda \cdot \vec{\nabla} g$.

Example 12.8.7, cont...)

This realization gives rise to the method of Lagrange Multipliers, which is focused on identifying values of x, y, and λ such that the three equations $\vec{\nabla} f(x,y) = \lambda \cdot \vec{\nabla} g(x,y)$ and $g(x,y)=0$ hold true. In this case, we have

Equation 1: $f_x(x,y) = 2x - 2 = \lambda \cdot 2x = \lambda \cdot g_x(x,y)$

Equation 2: $f_y(x,y) = 2y + 2 = \lambda \cdot 2y = \lambda \cdot g_y(x,y)$

Equation 3: $g(x,y) = x^2 + y^2 - 4 = 0$

To solve these 3 equations, we eliminate variables. We start with equation 1:

$$2x - 2 = \lambda \cdot 2x \quad \Rightarrow \quad 2x - \lambda \cdot 2x = 2$$

$$\Rightarrow \quad 2x \cdot (1 - \lambda) = 2$$

$$\Rightarrow \quad x \cdot (1 - \lambda) = 1$$

$$\Rightarrow \quad 1 - \lambda = \frac{1}{x}$$

$$\Rightarrow \quad \lambda = 1 - \frac{1}{x} = \frac{x-1}{x}.$$

Example 12.8.7...)

Now, we can substitute this expression for λ into equation 2 to find that if $\lambda = \frac{x-1}{x}$, then

$$2y + 2 = \lambda \cdot 2y \Rightarrow 2y + 2 = \frac{(x-1)}{x} \cdot 2y$$

$$\Rightarrow x \cdot (2y+2) = 2y \cdot (x-1)$$

$$\Rightarrow 2xy + 2x = 2xy - 2y$$

$$\Rightarrow 2x = -2y$$

$$\Rightarrow x = -y.$$

Finally, we can substitute $x = -y$ into equation 3 to find that

$$x^2 + y^2 - 4 = 0 \Rightarrow (-y)^2 + y^2 = 4$$

$$\Rightarrow 2y^2 = 4$$

Example 12.8.7...)

$$\Rightarrow \quad y^2 = 2$$

$$\Rightarrow \quad \sqrt{y^2} = \sqrt{2}$$

$$\Rightarrow \quad |y| = \sqrt{2}$$

$$\Rightarrow \quad y = -\sqrt{2} \quad \text{or} \quad y = +\sqrt{2}$$

$$\Rightarrow \quad (\sqrt{2}, -\sqrt{2}) \quad \text{or} \quad (-\sqrt{2}, \sqrt{2})$$

are points that satisfy the equation

$$\vec{\nabla} f(a,b) = \lambda \cdot \vec{\nabla} g(a,b)$$

We now test our objective function $f(x,y)$ at each of these points to find:

$$f(\sqrt{2}, -\sqrt{2}) = (\sqrt{2})^2 + (-\sqrt{2})^2 - 2\cdot\sqrt{2} - 2\sqrt{2} + 5$$

$$= 2 + 2 + 5 - 4\sqrt{2} = \boxed{9 - 4\sqrt{2}}$$ ← min $f(x,y)$ s.t. $g(x,y)=0$

$$f(-\sqrt{2}, \sqrt{2}) = (-\sqrt{2})^2 + (\sqrt{2})^2 + 2\sqrt{2} + 2\sqrt{2} + 5$$

$$= 2 + 2 + 5 + 4\sqrt{2} = \boxed{9 + 4\sqrt{2}}$$ ← max $f(x,y)$ s.t. $g(x,y)=0$

Procedure. *Method of Lagrange Multipliers in Two Variables* p. *953*

Let the objective function $f : \mathbb{R}^2 \to \mathbb{R}$ and the constraint function $g : \mathbb{R}^2 \to \mathbb{R}$ be differentiable on a region in $\mathbb{R}^2$. Assume that $\nabla g(x, y) \neq \mathbf{0}$ on the curve C defined by the equation $g(x, y) = 0$. To locate the maximum or minimum values of $f(x, y)$ subject to the constraint $g(x, y) = 0$, we carry out the following steps:

1. Find all the values of x, y and λ (if they exist) that satisfy the equations:

$$\nabla f(x, y) = \lambda \nabla g(x, y) \qquad \text{and} \qquad g(x, y) = 0.$$

2. Among the values of (x, y) found in Step 1, select the largest and smallest corresponding function values $f(x, y)$. These values are the maximum and minim values of f subject to the given constraint.

Example 12.9.1 p. 953)

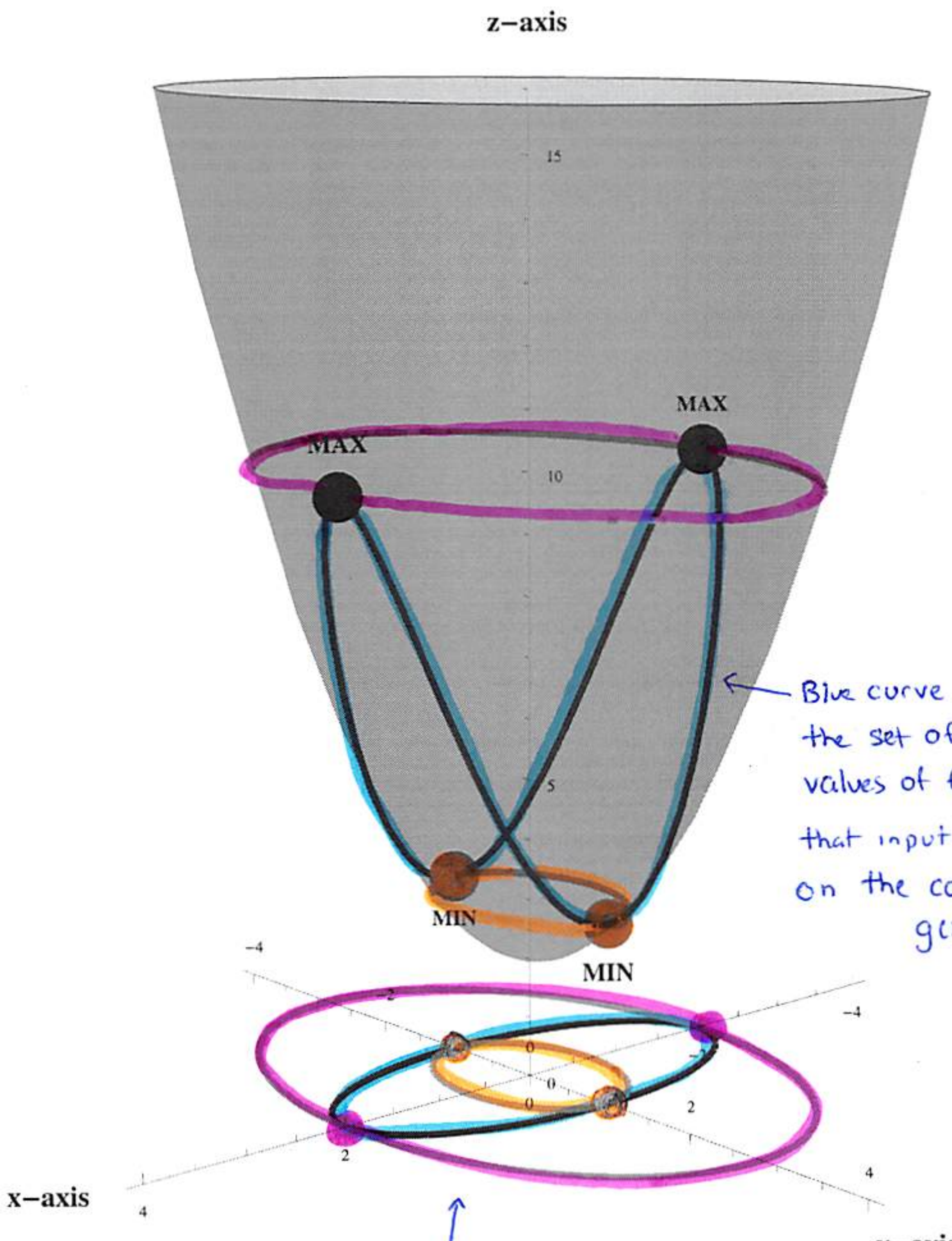
Find the maximum and minimum values of objective function

$$f(x,y) = 2x^2 + y^2 + 2$$

where x and y are restricted to be on the constraint curve C given by

$$g(x,y) = x^2 + 4y^2 - 4 = 0$$

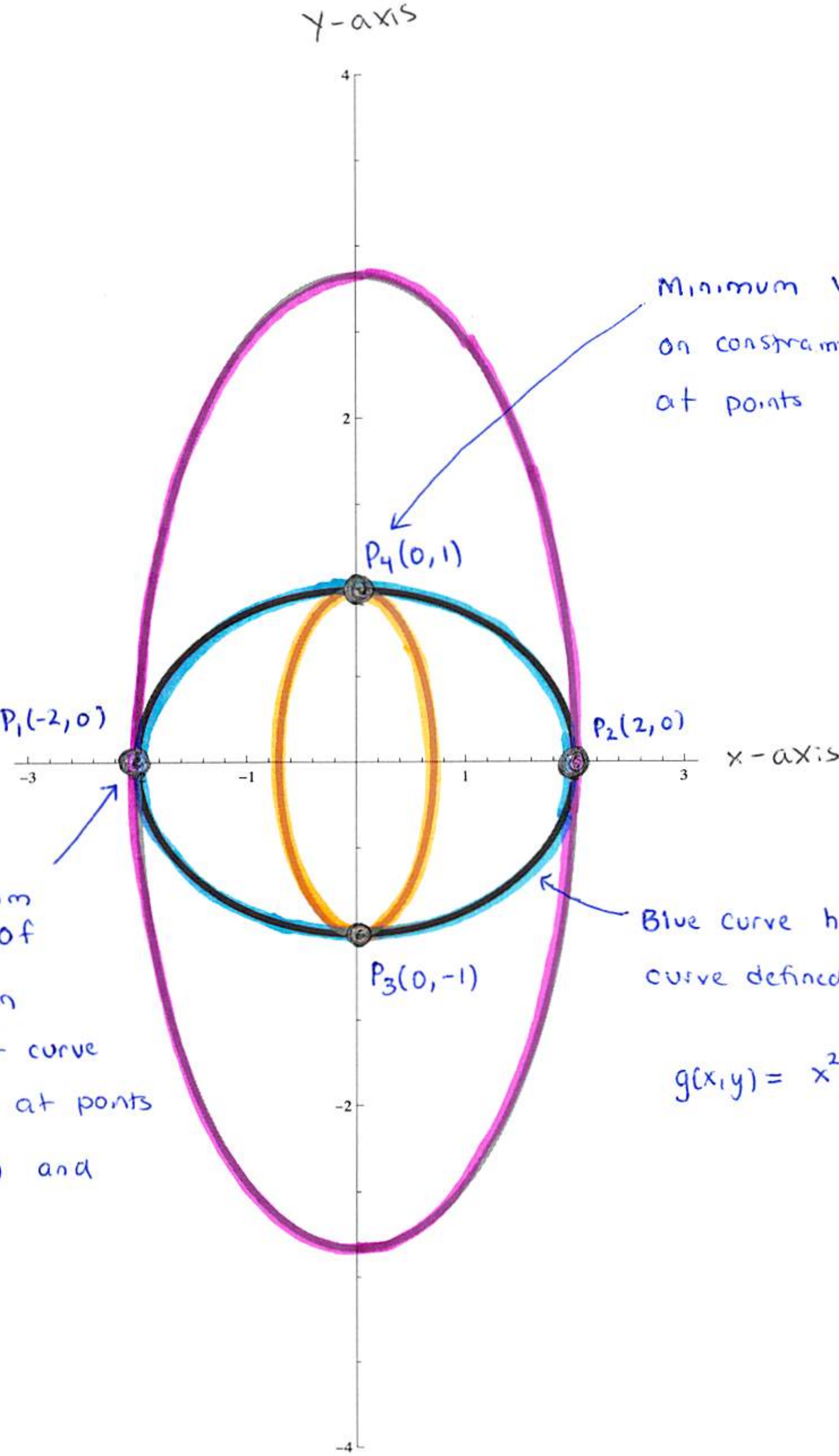
Surface Plot:
$f(x,y) = 2x^2+y^2+2$



Blue curve here is the set of all output values of $f(x,y)$ such that input (x,y) are on the constrant curve $g(x,y) = 0$

Here we embed the level curves of $f(x,y)$ that are tangent to the constrant curve $g(x,y)=0$ in $\mathbb{R}^3$ to visualize the geometry (see next pg for $\mathbb{R}^2$ image)

$\boxed{\text{L14, p.16}}$



Minimum values of $f(x,y)$ on constraint curve C occur at points $P_3(0,-1)$ and $P_4(0,1)$

Maximum values of $f(x,y)$ on constraint curve C occur at points $P_1(-2,0)$ and $P_2(2,0)$

Blue curve here is the constraint curve defined by equation

$$g(x,y) = x^2 + 4y^2 - 4 = 0$$

Example 12.9.1 p. 954 ...)

Next, lets apply our method of Lagrange Multipliers in Two Variables. We note our desired three equations are given by

Equation 1: $f_x(x,y) = 4x = \lambda \cdot 2x = \lambda \cdot g_x(x,y)$

Equation 2: $f_y(x,y) = 2y = \lambda \cdot 8y = \lambda \, g_y(x,y)$

Equation 3: $g(x,y) = x^2 + 4y^2 - 4 = 0$

Recall: Equations 1 & 2 result from the condition that $\vec{\nabla} f$ is parallel to $\vec{\nabla} g$, which is given as the vector equation

$$\vec{\nabla} f(x,y) = \lambda \cdot \vec{\nabla} g(x,y)$$

On the other hand, equation 3 is the constraint.

Now, let's do a little mathematical analysis on all three equations in order to simplify and solve.

Let's start with Equation 1 :

$$4x = 2x\lambda \implies 2x = x\lambda$$

$$\implies x \cdot (2 - \lambda) = 0$$

$$\implies x = 0 \quad \text{OR} \quad \lambda = 2$$

Now, let's consider each of these possibilities as a separate case and use the other two equations to find all points on curve C that satisfy our condition that $\vec{\nabla} f = \lambda \vec{\nabla} g$.

$\boxed{\text{Case } x = 0:}$

If $x = 0$, we know by equation 3 that

$$x^2 + 4y^2 - 4 = 0 \implies 4y^2 = 4$$

$$\implies y^2 = 1$$

$$\implies \sqrt{y^2} = |y| = 1$$

$$\implies y = +1 \quad \text{or} \quad y = -1$$

Then we have two candidates : $(0,1)$ or $(0,-1)$.

$\boxed{\text{Case } \lambda = 2:}$

By equation 2, we have

$$2y = 8y \cdot \lambda \Rightarrow 2y = 16y$$

$$\Rightarrow 14y = 0$$

$$\Rightarrow y = 0$$

Then, substituting $y=0$ into equation 3, we have

$$x^2 + 4y^2 - 4 = 0 \Rightarrow x^2 - 4 = 0$$

$$\Rightarrow x^2 = 4$$

$$\Rightarrow \sqrt{x^2} = |x| = 2$$

$$\Rightarrow x = +2 \quad \text{or} \quad x = -2$$

Then, we have another two candidates $(2,0)$ or $(-2,0)$.

Now we simply need to check the values of $f(x,y)$ at each of these points on C and choose the maximum and minimum values.

Point	Value of $f(x,y)$	Conclusion
$(0, 1)$	$f(0,1) = 3$	Min of $f(x,y)$ on curve C
$(0, -1)$	$f(0,-1) = 3$	min of $f(x,y)$ on curve C
$(2, 0)$	$f(2,0) = 10$	max of $f(x,y)$ on curve C
$(-2, 0)$	$f(-2,0) = 10$	max of $f(x,y)$ on curve C.

Thus, the minimum value of $f(x,y)$ subject to constraint $g(x,y)=0$ is $\boxed{3}$.

The maximum value of $f(x,y)$ subject to constraint $g(x,y)=0$ is $\boxed{10}$.

Procedure. *Method of Lagrange Multipliers in Three Variables* p. 955

Let the objective function $f : \mathbb{R}^3 \to \mathbb{R}$ and the constraint function $g : \mathbb{R}^3 \to \mathbb{R}$ be differentiable on a region in $\mathbb{R}^2$. Assume that $\nabla g(x, y, z) \neq \mathbf{0}$ on the curve C defined by the equation $g(x, y, z) = 0$. To locate the maximum or minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = 0$, we carry out the following steps:

1. Find all the values of x, y, z, and λ (if they exist) that satisfy the equations:

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \qquad \text{and} \qquad g(x, y, z) = 0.$$

2. Among the values of (x, y, z) found in Step 1, select the largest and smallest corresponding function values $f(x, y, z)$. These values are the maximum and minim values of f subject to the given constraint.

Again we notice that the equation

$$\vec{\nabla} f(x,y,z) = \lambda \vec{\nabla} g(x,y,z)$$

is a vector equation. Thus, when using the method of Lagrange Multipliers in three variables, we solve four equations in four unknowns:

Equation 1: $f_x(x,y,z) = \lambda g_x(x,y,z)$

Equation 2: $f_y(x,y,z) = \lambda g_y(x,y,z)$

Equation 3: $f_z(x,y,z) = \lambda g_z(x,y,z)$

Equation 4: $g(x,y,z) = 0$

Note: Equations 1, 2 & 3 result from $\vec{\nabla} f = \lambda \vec{\nabla} g$.
The unknowns in these equations are x, y, z, λ.

Example 12.9.2 p. 955)

Find the minimum distance between the point $P(3,4,0)$ and the surface of cone $z^2 = x^2 + y^2$.

Solution 1: Method of Lagrange Multipliers in 3-variables

Let's begin by translating our problem into our desired general form. In this case, we want to minimize the distance between a point (x,y,z) and point $P(3,4,0)$. This distance is given by

$$d(x,y,z) = \sqrt{(x-3)^2 + (y-4)^2 + z^2}$$

Though this is the function we will eventually minimize, trying to find $\vec{\nabla} d$ will be algebraically exhausting. So, instead, let's simplify our work by defining our objective function to be d^2 and workng to

$$\min_{(x,y,z)\in\mathbb{R}^3} f(x,y,z) = \min_{(x,y,z)\in\mathbb{R}^3} (x-3)^2 + (y-4)^2 + z^2$$

Subject to the constraint that (x,y,z) are on the cone

or

subject to $\quad g(x,y,z) = z^2 - x^2 - y^2 = 0$

Now, we have a Lagrange Multiplier problem in our desired form.

Now, let's apply our method of Lagrange Multipliers in Three variables. We note our desired Four equations are given by:

Equation 1: $f_x(x,y,z) = 2(x-3) = \lambda\cdot(-2x) = \lambda\, g_x(x,y,z)$

Equation 2: $f_y(x,y,z) = 2\cdot(y-4) = \lambda\cdot(-2y) = \lambda\, g_y(x,y,z)$

Equation 3: $f_z(x,y,z) = 2z = \lambda(2z) = \lambda\, g_z(x,y,z)$

Equation 4: $g(x,y,z) = z^2 - x^2 - y^2 = 0$

Again, we see equations 1, 2, & 3 result from the condition that

$$\vec{\nabla} f(x,y,z) = \lambda \vec{\nabla} g(x,y,z)$$

and equation 4 is simply the constraint.

Now that we have all four equations, let's do some analysis to solve for all points on C that satify the condition that $\vec{\nabla} f = \lambda\cdot\vec{\nabla} g$.

Since Equation 3 is the simplest, let's start here:

$$2z = 2z \cdot \lambda \Rightarrow 2z(1-\lambda) = 0$$

$$\Rightarrow z = 0 \quad \text{OR} \quad \lambda = 1$$

Again, let's consider each case separately.

$\boxed{\text{Case } z = 0}$ ← any value of λ allowed

If $z=0$, we know by equation 4 that

$$z^2 - x^2 - y^2 = 0 \Rightarrow -x^2 - y^2 = 0$$

$$\Rightarrow x^2 = -y^2$$

$$\Rightarrow x = 0 \quad \text{and} \quad y = 0$$

But, substituting $x=0$ into equation 1, we see

$$2(x-3) = -2x \cdot \lambda \Rightarrow 2 \cdot (x-3) = 0$$

$$\Rightarrow x = 3 \quad \Rightarrow\Leftarrow$$

Similarly for $y=0$. Thus no solution results in this case.

$\boxed{\text{Case } \lambda = 1 \quad \text{AND} \quad z \neq 0}$ ← we just saw for any value of λ, $z = 0$ does not produce a solution

If $\lambda = 1$, we substitute this into equation 1 to find

$$2\cdot(x-3) = -2\cdot x\cdot\lambda \Rightarrow 2x - 6 = -2x$$

$$\Rightarrow 4x = 6$$

$$\Rightarrow x = 3/2$$

We repeat this process for $\lambda = 1$ in equation 2

$$2\cdot(y-4) = -2y\lambda \Rightarrow 2y - 8 = -2y$$

$$\Rightarrow 4y = 8$$

$$\Rightarrow y = 2$$

Now, we use $x = 3/2$ and $y = 2$ in equation 4 to find

$$z^2 - x^2 - y^2 = 0 \Rightarrow z^2 = \frac{9}{4} + 4$$

$$\Rightarrow \quad z^2 = \frac{25}{4}$$

$$\Rightarrow \quad \sqrt{z^2} = |z| = \frac{5}{2}$$

$$\Rightarrow \quad z = +\frac{5}{2} \qquad \text{or} \qquad z = -\frac{5}{2}$$

This results in two candidates on surface $g(x,y,z)=0$ that satisfy condition $\vec{\nabla} f = \lambda \cdot \vec{\nabla} g$, given as points

$$\left(\frac{3}{2}, 2, \frac{5}{2}\right) \qquad \text{AND} \qquad \left(\frac{3}{2}, 2, -\frac{5}{2}\right)$$

Moving away from either of these points has the effect of increasing the value of f compared with

$$f\left(\frac{3}{2}, 2, \frac{5}{2}\right) = \frac{25}{2} = f\left(\frac{3}{2}, 2, -\frac{5}{2}\right)$$

Thus, each of these points is a local minimum of $f(x,y,z)$

Since $f = d^2$, we know the least distance from our point $P(3,4,0)$ to the cone is $\sqrt{\frac{25}{2}} = \frac{5}{\sqrt{2}}$.