Math 1C: Calculus III

Lesson 16: Series

Reference: Brigg's "Calculus: Early Transcendentals, Second Edition"
Section 8.3: Infinite Series, p. 619 - 626

Definition. *p. 602* ***Infinite Series***

Let the sequence $\{a_n\}_{n=1}^{\infty}$ be given. The **sequence of partial sums** $\{S_n\}_{n=1}^{\infty}$ associated with this sequence has the terms

$$\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ S_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ S_n &= a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k \text{ for } n \in \mathbb{N} \end{aligned}$$

The **infinite series** associated with this sequence is the limit of the sequence of partial sums

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$$

Example 1

Let $a_n = n$.

Consider the sequence of partial sums associated with this sequence

$$S_n = \sum_{k=1}^{n} a_k = 1 + 2 + 3 + \cdots + n$$

Quick side note:

$$S_n + S_n = (1 + 2 + \cdots + n) + (n + n-1 + \cdots + 1)$$
$$= (n+1) + (n+1) + \cdots + (n+1)$$
$$= n \cdot (n+1)$$

$$\Rightarrow 2S_n = (n+1)n$$

$$\Rightarrow S_n = \frac{(n+1) \cdot n}{2}$$

Thus we see

$$S_n = \frac{n(n+1)}{2}$$

- What happens as n gets really large?

- $S_n \to \infty$ as $n \to \infty$

Divergent Series : $\sum_{n=1}^{\infty} a_n$

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Definition. *p.602* ***Limit of an Infinite Series***

If the sequence of partial sums $\{S_n\}_{n=1}^{\infty}$ approaches a limit L, we say

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} S_n = L.$$

If $\lim_{n\to\infty} S_n = L$, then we say the infinite series **converges** to L.

If the sequence of partial sums diverges, we say the infinite series **diverges**.

Example 8.3. Quick Check 3 p. 621

Let $a_n = \frac{1}{2^n}$

consider the sequence of partial sums

$$S_n = \sum_{k=1}^{n} a_n = \sum_{k=1}^{n} \frac{1}{2^k}$$

side Note: Closed form Representation of S_n

$S_1 = \boxed{\frac{1}{2}}$

$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{2}{4} + \frac{1}{4} = \boxed{\frac{3}{4}}$

$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{3}{4} + \frac{1}{8} = \frac{6}{8} + \frac{1}{8} = \boxed{\frac{7}{8}}$

$S_4 = S_3 + \frac{1}{16} = \frac{7}{8} + \frac{1}{16} = \frac{14+1}{16} = \frac{15}{16}$

$S_5 = \qquad = \frac{31}{32}$

$\vdots$

$S_n = S_{n-1} + \frac{1}{2^n} = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}$

Test to see:

Thus, $\underbrace{\sum_{k=1}^{\infty} a_k}_{\text{Convergent Series}} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{2^n} = 1 - \lim_{n\to\infty} \frac{1}{2^n} = 1.$

Definition. *p. 620* ***Geometric Sum Formula***

A **geometric sum** with n terms has the form

$$S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} = \sum_{k=1}^{n} ar^{k-1}$$

Using this formulation, we see that

$$S_n - rS_n = a - ar^n.$$

If $r \neq 1$, we can solve for S_n and produce the general formula for the geometric sum

$$S_n = \boxed{\sum_{k=1}^{n} ar^{k-1} = a\frac{1-r^n}{1-r}}$$

Note: We can write the series using different upper and lower index as follows:

$$\sum_{k=1}^{n} ar^{k-1} = \sum_{k=0}^{n-1} ar^k.$$

Definition of Geometric Series p. 620

An important example of an infinite series is called the geometric series.

Let $a_n = a \cdot r^{n-1}$ for $a \neq 0$ and $r \in \mathbb{R}$.

Then consider the sequence of partial sums

$$S_n = \sum_{k=1}^{n} a_k$$

$$= \sum_{k=1}^{n} a \cdot r^{k-1}$$

$$= a + ar^1 + ar^2 + a \cdot r^3 + \cdots + a \cdot r^{n-1}$$

$$= a(1 + r + r^2 + r^3 + \cdots + r^{n-1})$$

$$= a \cdot \left[\sum_{k=1}^{n} r^{k-1} \right]$$

- refer back to warm up problem on generalized formula for difference of nth powers

$$\boxed{\stackrel{*}{=} a \cdot \frac{1 - r^n}{1 - r}}$$

- What happens to this sequence as r takes on different values for a fixed $a \in \mathbb{R} - \{0\}$ $(a \neq 0)$?

If $r=1$: $\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} a \cdot r^{k-1} = a \sum_{k=1}^{n} r^{k-1} = a \cdot \sum_{k=1}^{n} 1^{k-1} = a \cdot n$

$$\Rightarrow \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} a \cdot n = +\infty$$

divergent series

if $-1 < r < 1$: $\lim_{n\to\infty} r^n = 0$

$$\Rightarrow \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a \cdot r^{k-1}$$

$$= \lim_{n\to\infty} \frac{a(1-r^n)}{1-r}$$

$$= a \cdot \lim_{n\to\infty} \frac{1-r^n}{1-r}$$

$$= a \cdot \frac{1}{1-r}$$

If $r \le -1$ or $r > 1$; series is divergent

Geometric Series: $\sum_{n=1}^{\infty} a r^{n-1} = a + a \cdot r + a r^2 + a r^3 + \cdots$

is convergent if $|r| < 1$ and it's sum is

$$\sum_{n=1}^{\infty} a r^{n-1} = \frac{a}{1-r} \qquad |r| < 1$$

If $|r| \ge 1$, the geometric series is divergent.

Theorem 8.7. *p. 621* ***Geometric Series Test***

Let $a \neq 0$ and let r be a real number. Then, the series

$$\sum_{k=1}^{\infty} ar^{k-1}$$

has the following convergence behavior:

If $|r| < 1$, then the series converges and $\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}$.

If $|r| \geq 1$, then the series diverges.

Note: We can write the series using different upper and lower index as follows:

$$\sum_{k=1}^{\infty} ar^{k-1} = \sum_{k=0}^{\infty} ar^{k}$$

In both cases, we can determine the convergence behavior based on the geometric sum formula combined with our knowledge of the limits of geometric sequences.

Example 8.3.1a) Is the series

p. 621

$$\sum_{n=1}^{\infty} 2^{2n} \cdot 3^{1-n}$$

convergent or divergent

Solution: Recall general form of geometric series

$$S_n = \sum_{K=1}^{n} a \cdot r^{K-1}$$

$$= a + ar + ar^2 + \cdots + ar^{n-1}$$

In this case, we have

$$a_n = 2^{2n} \cdot 3^{1-n}$$

$$= 4^n \cdot \frac{3}{3^n}$$

$$= 3 \cdot \left[\frac{4}{3}\right]^n$$

$$= 3 \cdot \frac{4}{3} \cdot \left[\frac{4}{3}\right]^{n-1}$$

$$= 4 \cdot \left[\frac{4}{3}\right]^{n-1}$$ let $a = 4$, $r = \frac{4}{3}$

Then we see

$$S_n = \sum_{k=1}^{n} a_n$$

$$= \sum_{k=1}^{n} 4 \cdot \left[\frac{4}{3}\right]^{k-1}$$

$$= a \left[\frac{1 - [4/3]^n}{1 - 4/3} \right]$$

Since $\frac{4}{3} > 1$, we know by previous result this series is divergent!

Example 8.3.1 c p. 621)

Evaluate the following geometric series or state that the series diverges

$$\sum_{K=2}^{\infty} 3(-0.75)^K$$

Solution: Our given series has a similar form to a geometric series. However, in order to apply our Geometric Series Test, we need to translate this series into general form

Recall: In the statement of the geometric series test, we expect series in the form

$$\sum_{n=1}^{\infty} a \cdot r^{n-1}$$

To this end, consider

$$\sum_{K=2}^{\infty} 3 \cdot (-0.75)^K = \sum_{K=2}^{\infty} 3 \cdot \left[-\frac{3}{4}\right]^K$$

Example 8.3.1 c p. 621 ...

$$= 3 \cdot \left[\frac{-3}{4}\right]^2 + 3 \cdot \left[\frac{-3}{4}\right]^3 + 3 \cdot \left[\frac{-3}{4}\right]^4 + 3 \cdot \left[\frac{-3}{4}\right]^5 + \cdots$$

$$= 3 \cdot \left[\frac{-3}{4}\right]^2 \cdot \left(1 + \left[\frac{-3}{4}\right]^1 + \left[\frac{-3}{4}\right]^2 + \left[\frac{-3}{4}\right]^3 + \cdots\right)$$

$$= 3 \cdot \left[\frac{-3}{4}\right]^2 \cdot \sum_{n=1}^{\infty} \left[\frac{-3}{4}\right]^{n-1}$$

← let $r = -3/4$ and apply geometric series test here: notice $|r| < 1$ so we know the series converges

$$= 3 \cdot \left[\frac{-3}{4}\right]^2 \cdot \frac{1}{1 - -3/4}$$

$$= 3 \cdot \frac{9}{16} \cdot \frac{1}{1 + 3/4}$$

$$= \frac{27}{16} \cdot \frac{4}{7} = \boxed{\frac{27}{28}}$$

Similar exercise to Example 8.3.2 p.622) Expressing a repeating decimal number as a ratio of integers

Write the decimal number

$$2.3\overline{17} = 2.3171717\ldots$$

as a ratio of integers.

Solution: Consider

$$2.3171717\ldots = 2.3 + \frac{17}{10^3} + \frac{17}{10^5} + \frac{17}{10^7} + \cdots$$

Let's focus our attention on the series

$$\sum_{n=1}^{\infty} \frac{17}{10^{2n+1}} \rightarrow$$ want to get this into general form $\rightarrow \sum_{n=1}^{\infty} a \cdot r^{n-1}$

Recall: Geometric series

Consider $a_n = \dfrac{17}{10^{2n+1}}$

$$= \frac{17}{10 \cdot 10^{2n}}$$

Note: $10^{2n} = [10^2]^n = 100^n$

$$= \frac{17}{10 \cdot 100^n}$$

$$= \frac{17}{10^3 \cdot (100)^{n-1}} = \frac{17}{10^3} \cdot \left[\frac{1}{100}\right]^{n-1}$$

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Similar exercise to
Example 8.3.2 p. 622 ...

Then we can write

$$2.31\overline{17} = 2.3 + \sum_{n=1}^{\infty} \frac{17}{10^3} \cdot \left[\frac{1}{100}\right]^{n-1}$$

with $a = \dfrac{17}{10^3}$ and

ratio $r = \dfrac{1}{100}$

$$= 2.3 + \left[\frac{17}{10^3} \cdot \frac{1}{1 - \frac{1}{100}}\right]$$

$$= \frac{23}{10} + \frac{17}{10^3} \cdot \frac{1}{\frac{99}{100}}$$

$$= \frac{23}{10} + \frac{17}{10^3} \cdot \frac{10^2}{99}$$

$$= \frac{23}{10} + \frac{17}{990}$$

Check using TI calc ✓

$$= \frac{23 \cdot 99}{990} + \frac{17}{990} = \frac{2277 + 17}{990} = \boxed{\frac{2294}{990}}$$

Example 8.3.3b p. 622) Telescoping Sum

Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

Converges and find the sum.

Solution: First notice this is NOT a geometric series.

Consider $a_n = \frac{1}{n(n+1)}$ and define the sequence of partial sums

$$S_n = \sum_{k=1}^{n} a_k$$

$$= \sum_{k=1}^{n} \frac{1}{n(n+1)}$$

$$= \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n \cdot (n+1)}$$

This is not immediately clear how to deal with this.

Recall partial fraction decomposition

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

$$\Rightarrow \quad 1 = A(n+1) + Bn$$

$$\Rightarrow \quad 1 = A \cdot n + A + B \cdot n$$

$$\Rightarrow \quad 1 = (A+B)n + A$$

$$\Rightarrow \quad A + B = 0 \qquad A = 1$$

$$\Rightarrow \quad B = -1$$

$$\Rightarrow \quad \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

Then $S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{k+1}$

$$= \underbrace{\left[\frac{1}{1} - \frac{1}{2}\right]}_{k=1} + \underbrace{\left[\frac{1}{2} - \frac{1}{3}\right]}_{k=2} + \underbrace{\left[\frac{1}{3} - \frac{1}{4}\right]}_{k=3} + \cdots + \underbrace{\left[\frac{1}{n} - \right.}_{k=n}$$

$$= 1 + \left[-\frac{1}{2} + \frac{1}{2}\right] + \left[-\frac{1}{3} + \frac{1}{3}\right] + \cdots + \left[\frac{-1}{n} + \frac{1}{n}\right] + \frac{-1}{n+1}$$

$$= 1 - \frac{1}{n+1}$$

Thus we have

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{n+1}$$

$$= 1 - \lim_{n\to\infty} \frac{1}{n+1}$$

$$\boxed{= 1.}$$

> This is a famous example known as telescoping sums. The technique is similar to the partial fraction techiques of integrals discussed in Section 7.5 p. 541 – 551

Note: A good way to test whether to use the telescoping sum technique is to write out the first few terms of the sequence of partial sums and see if anything "cancels out".

If so, try a partial fraction decomposition.

Sometimes, we may need to do some creative algebra before it's clear that this technique will work.

Definition. ***Telescoping Series Technique***

A **telescoping series** is an infinite series whose partial sums eventually only have a fixed number of terms after cancellation. To apply this technique, we need to write each term of the series as a difference of terms from a a specially designed sequence. In particular, if we want to apply the telescoping series technique to series

$$\sum_{n=1}^{\infty} b_n$$

then, we need to write $b_n = a_n - a_{n+1}$ for a properly constructed sequence $\{a_n\}_{n=1}^{\infty}$. If we can do this, then we can use the method of differences will generally lead to analysis that follows the pattern below:

$$\sum_{n=1}^{\infty} b_n = \lim_{N\to\infty} \sum_{n=1}^{N} \Big[\, a_n - a_{n+1} \,\Big]$$

$$= \lim_{N\to\infty} \left(\Big[\, a_1 - a_2 \,\Big] + \Big[\, a_2 - a_3 \,\Big] + \Big[\, a_3 - a_4 \,\Big] + \cdots + \Big[\, a_{N-1} - a_N \,\Big]\right)$$

$$= \lim_{N\to\infty} \left(a_1 + \Big[-a_2 + a_2 \,\Big] + \Big[-a_3 + a_3 \,\Big] \cdots + \Big[-a_{N-1} + a_{N-1} \,\Big] - a_N\right)$$

$$= \lim_{N\to\infty} \; a_1 - a_N$$

$$= a_1 - \lim_{N\to\infty} a_N$$

Notice that, in the method of differences, the second part of each difference cancels the first part of the next term. The original series converges if and only the limit $\lim\limits_{N\to\infty} a_N$ exists.

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Exercise 8.3.67 p. 624) Find the limit of the following infinite series

$$\sum_{k=0}^{\infty} \frac{1}{16k^2 + 8k - 3}$$

Solution: We note this is not applicable for the geometric series test. We also might try the factoring the denominator:

The AC Method

Multiply

-48

$+12$ $\quad -4$

8

Add

$\Rightarrow 8k = 12k - 4k$

$$16k^2 + 8k - 3$$

$a = 16 \qquad b = 8 \qquad c = -3$

$$= 16k^2 + 12k - 4k - 3$$

$$= 4k(4k+3) - 1(4k+3)$$

$$= (4k-1)\cdot(4k+3)$$

Using this factorization, let's try a partial fraction decomposition

$$\frac{1}{(4k-1)\cdot(4k+3)} = \frac{A}{4k-1} + \frac{B}{4k+3}$$

$\Rightarrow \quad 1 = A \cdot (4K+3) + B(4K-1)$

$\Rightarrow \quad 1 = (4A+4B)K + 3A - B$

$\Rightarrow \quad 4A + 4B = 0$ and $3A - B = 1$

$\Rightarrow \quad A = -B$ AND $3A = 1 + B$

$\Rightarrow \quad -3B = 1 + B$

$\Rightarrow \quad B = -1/4$ and $A = 1/4$

$$\Rightarrow \sum_{K=0}^{\infty} \frac{1}{(4K-1)(4K+3)} = \sum_{K=0}^{\infty} \frac{+1}{4(4K-1)} + \frac{-1}{4(4K+3)}$$

$$= \underbrace{-\frac{1}{4} + -\frac{1}{12}}_{K=0} + \underbrace{\frac{1}{12} - \frac{1}{28}}_{K=1} + \frac{1}{28} - \frac{1}{44} + \cdots$$

Here we see evidence of Telescoping sum technique

$$= \lim_{N \to \infty} \sum_{K=0}^{N} \frac{1}{4 \cdot (4K-1)} - \frac{1}{4(4K+3)}$$

To find limit, consider

$$\sum_{K=0}^{\infty} \frac{1}{4(4K-1)} - \frac{1}{4(4K+3)}$$

$$= \lim_{N\to\infty} \sum_{K=0}^{N} \frac{1}{(4K-1)} - \frac{1}{(4K+3)}$$

$$= \lim_{N\to\infty} \frac{1}{4} \sum_{K=0}^{N} \frac{1}{4K-1} - \frac{1}{4K+3}$$

$$= \frac{1}{4}\left[\lim_{N\to\infty} -1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{7} + \frac{1}{7} + \cdots + \frac{1}{4N-1} - \frac{1}{4N+3}\right.$$

$$= \frac{1}{4} \lim_{N\to\infty} \left[-1 - \frac{1}{4N+3}\right]$$

$$= \frac{1}{4} \lim_{N\to\infty} \left[-\frac{(4N+3)-1}{4N+3}\right] \rightarrow = -\frac{4}{4} \cdot \lim_{N\to\infty} \frac{N+1}{4N+3}$$

$$= \frac{1}{4} \lim_{N\to\infty} \frac{-4N-4}{4N+3} \qquad = -1 \cdot +\frac{1}{4} = \boxed{-\frac{1}{4}}$$

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