Math 1C: Calculus III

Lesson 17: Divergence and Integral Tests

Reference: Brigg's "Calculus: Early Transcendentals, Second Edition"
Topics: Section 8.4: The Divergence and Integral Tests, p. 627 - 640

Theorem 8.8. *p. 627* ***Divergence Test***

If the infinite series $\sum a_k$ converges, then $\lim_{k\to\infty} a_k = 0$.

Equivalently, if $\lim_{k\to\infty} a_k \neq 0$, then the infinite series $\sum a_k$ diverges.

Note: The divergence test cannot be used to conclude that a series converges. This test ONLY provides a quick mechanism to check for divergence.

Proof: Let $S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$.

Notice $a_n = S_n - S_{n-1}$

$= (a_1 + a_2 + \cdots + a_n) - (a_1 + a_2 + \cdots + a_{n-1})$.

If $\sum_{k=1}^{\infty} a_k = S$ exists, then $\lim_{n\to\infty} S_n = S = \lim_{n\to\infty} S_{n-1}$

Then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1})$

$= \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$

$= S - S = 0$ ✓

Remark 1: With any series $\sum_{n=1}^{\infty} a_n$, we associate two sequences

1. sequence of partial sums : $\{S_n\}_{n=1}^{\infty}$ where

$$S_n = \sum_{k=1}^{n} a_k$$

2. sequence of terms : $\{a_n\}_{n=1}^{\infty}$

Thm 6 states: If $\sum_{n=1}^{\infty} a_n$ converges, then

the limit of sequences $\{S_n\}_{n=1}^{\infty}$ is S

and the limit of sequence $\{a_n\}_{n=1}^{\infty}$ is 0.

Remark 2: The converse of Theorem 8.2.6 is NOT TRUE

If $\lim_{n\to\infty} a_n = 0$, we cannot conclude that

$\sum_{n=1}^{\infty} a_n$ is convergent.

Harmanic series: $\boxed{\sum_{n=1}^{\infty} \frac{1}{n}}$

Test For Divergence p. 628

If $\lim_{n\to\infty} a_n \neq 0$ (or DNE), then the

series $\sum_{n=1}^{\infty} a_n$ is divergent.

Similar to

Example 8.4.1a p. 628

Show that the series $\sum_{n=1}^{\infty} \left[\frac{n^2}{5n^2+4}\right]$ diverges.

Solution: Let $a_n = \frac{n^2}{5n^2+4}$ be the sequence of terms

For our series,

Consider $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{5n^2+4}$

$$= \lim_{n\to\infty} \frac{n^2}{5n^2+4} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{1}{5+4/n^2}$$

$$= \frac{1}{5} \neq 0$$

By Divergence Test, $\sum_{n=1}^{\infty} a_n$ DNE.

EXAMPLE: $\lim_{n\to\infty} a_n$ & $\lim_{n\to\infty} S_n$ DN

Example 8.4.1c p. 628

Show the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

is divergent.

Solution: We will use a "squeeze thm"-esc proof by bounding $\sum_{n=1}^{\infty} \frac{1}{n}$ below by a series that diverges. Let $S_n = \sum_{k=1}^{n} \frac{1}{k}$

Consider

$$S_2 = \sum_{k=1}^{2} \frac{1}{k} = \boxed{1 + \frac{1}{2}}$$

$$S_4 = \sum_{k=1}^{4} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$$

$$> 1 + \frac{1}{2} + \left[\frac{1}{4} + \frac{1}{4}\right] = 1 + \frac{1}{2} + \frac{1}{2} = \boxed{1 + \frac{2}{2}}$$

$$S_8 = \sum_{k=1}^{8} \frac{1}{k} = 1 + \frac{1}{2} + \left[\frac{1}{3} + \frac{1}{4}\right] + \left[\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right]$$

$$> 1 + \frac{1}{2} + \left[\frac{1}{4} + \frac{1}{4}\right] + \left[\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right]$$

$$= 1 + \frac{1}{2} + \frac{2}{4} + \frac{4}{8} = \boxed{1 + \frac{3}{2}}$$

$$S_{16} = \sum_{k=1}^{16} \frac{1}{k} = 1 + \frac{1}{2} + \left[\frac{1}{3} + \frac{1}{4}\right] + \left[\frac{1}{5} + \cdots + \frac{1}{8}\right] + \left[\frac{1}{9} + \cdots + \frac{1}{16}\right]$$

$$> 1 + \frac{1}{2} + \left[\frac{1}{4} + \frac{1}{4}\right] + \left[\frac{1}{8} + \cdots + \frac{1}{8}\right] + \left[\frac{1}{16} + \cdots + \frac{1}{16}\right]$$

$$\boxed{= 1 + \frac{4}{2}}$$

Thus we see

$$S_{32} > 1 + \frac{5}{2}$$

$$S_{64} > 1 + \frac{6}{2}$$

$$S_{2^n} > 1 + \frac{n}{2}$$

$$\Rightarrow \lim_{n\to\infty} S_{2^n} \longrightarrow \infty \quad \text{as} \quad n \to \infty$$

Thus $\{s_n\}_{n=1}^{\infty}$ is divergent.

$\therefore$ The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Definition. *p. 628* ***Harmonic Series***

The famous **harmonic series** is given by $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$

Theorem 8.9. *p. 630* ***Harmonic Series***

The **harmonic series** $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, even though the terms of the series approach zero.

The Integral & Comparison Tests:

Exercise 8.4.66 p. 639

Consider the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots$$

This is Known as the Basel Problem

We can use mathematica to attempt to find a solution.

See our mathematica Notebook for today's Lesson.

- ☐ Demonstrate how to create series in mathematica
- ☐ Demonstrate how to create variable upper bound
- ☐ Demonstrate how to create a table of values in Mathematica
- ☐ Illustrate the concept behind the integral tests

Theorem 8.10. *p. 630* ***Integral Test***

Suppose the function $f(x)$ satisfies the following three conditions for $x \geq 1$:

i. $f(x)$ is continuous

ii. $f(x)$ is positive

iii. $f(x)$ is decreasing

Suppose also that $a_k = f(k)$ for all $k \in \mathbb{N}$. Then

$$\sum_{k=1}^{\infty} a_k \text{ and } \int_1^{\infty} f(x)dx$$

either both converge or both diverge. In the case of convergence, the value of the integral is NOT equal to the value of the series.

Note: The Integral Test is used to determine *whether* a series converges or diverges. For this reason, adding or subtracting a finite number of terms in the series or changing the lower limit of the integration to another finite point does not change the outcome of the test. For this reason, the Integral Test does NOT depend on the lower index of the series or the lower limit of the integral.

Similar to
Exercise 8.4.11 p. 638

Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ converges?

Solution: Let $a_n = \frac{\ln(n)}{n}$.

Define the function $f(x) = \frac{\ln(x)}{x} \Rightarrow a_n = f(n)$

Notice: ✓ $f(x)$ is continuous for $x > 1$

✓ $f(x)$ is positive for $x > 1$

$\ln(x) > 0 \Leftrightarrow e^{\ln(x)} > e^0$

$\Leftrightarrow x > 1$

– $f(x)$ is decreasing

$$f'(x) = \frac{1}{x^2} + \frac{-\ln(x)}{x^2} = \frac{1-\ln(x)}{x^2}$$

By decreasing test, we know $f(x)$ decreasing iff $f'(x) < 0$

$\Rightarrow \frac{1-\ln(x)}{x^2} < 0$

$\Rightarrow 1 - \ln(x) < 0$

$\Rightarrow 1 < \ln(x)$

$\Rightarrow e < x$

L17, p.9

Thus, we can apply the integral test

$$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{\ln(x)}{x} \cdot dx$$

$$= \int_1^\infty \ln(x) \cdot \frac{1}{x}\,dx$$

<u>U-substitution</u> : undo chain

Let $u(x) = \ln(x)$

$\Rightarrow du = \frac{1}{x}\,dx$

$$= \int_{x=1}^{x=\infty} u\,du$$

$$= \lim_{t\to\infty} \int_{x=1}^{x=t} u\,du$$

$$= \lim_{t\to\infty} \frac{u^2}{2}\Bigg|_{x=1}^{x=t}$$

$$= \lim_{t\to\infty} \frac{(\ln(x))^2}{2}\Bigg|_{x=1}^{x=t}$$

$$= \lim_{t\to\infty} \frac{t^2}{2} - \frac{0^2}{2} = \lim_{t\to\infty} \frac{t^2}{2} = +\infty$$

By the integral test, we see our series is divergent! $\boxed{\text{L17 p}}$

Thm 8.11 p. 632

For what values of p is the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

convergent ?

Solution: Let's consider different values of p:

I. $p<0$: (p Negative)

II. $p=0$: (p zero)

III. $p>0$: (p positive)

Case I. : $p<0$ (p negative)

If $p<0 \Rightarrow p=-r$ for some $r>0$

$$\Rightarrow \frac{1}{n^p} = \frac{1}{n^{-r}} = n^r \quad \text{for } r>0$$

$$\Rightarrow a_n = n^r$$

$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} n^r = +\infty$$

$\Rightarrow \sum_{n=1}^{\infty} a_n$ diverges by divergence test.

Case II: If $p=0 \Rightarrow \frac{1}{n^p} = \frac{1}{n^0} = \frac{1}{1} = 1$

$\Rightarrow a_n = 1$

$\Rightarrow \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} 1 = +\infty$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^p}$ divergent.

Case III: If $p>0 \Rightarrow a_n = \frac{1}{n^p}$ □ What can we do from here?

see example 7.8.2 p.572

Let $f(x) = \frac{1}{x^p}$. Notice ✓ $f(x)$ is continuous for $x \geq 1$

✓ $f(x)$ is positive for $x \geq 1$

✓ $f(x)$ is decreasing for $x \geq 1$

$f'(x) = \frac{-p}{x^{p+1}} = -1 \cdot \frac{p}{x^p} < 0$

$$\Rightarrow \int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^t x^{-p}\,dx$$

$$= \lim_{t\to\infty} \left. \frac{x^{-p+1}}{1-p} \right|_{x=1}^{t}$$

$$= \lim_{t\to\infty} \frac{t^{1-p}}{1-p} - \frac{1}{1-p}$$

$$= \lim_{t\to\infty} \frac{1}{c \cdot t^{p-1}} - \frac{1}{c} \qquad \text{where } c = 1-p$$

$$= \lim_{t\to\infty} \frac{1}{c}\left[\frac{1}{t^{p-1}} - 1\right]$$

If $0 < p < 1 \Rightarrow -1 < p-1 < 0 \Rightarrow 1 > 1-p > 0$

$$\Rightarrow \frac{1}{t^{p-1}} = t^{1-p}$$

$$\Rightarrow \lim_{t\to\infty} \frac{1}{t^{p-1}} = \lim_{t\to\infty} t^{1-p} = +\infty$$

$$\Rightarrow \int_1^\infty \frac{1}{x^p}\,dx \text{ diverges}$$

If $p = 1 \Rightarrow a_n = \frac{1}{n}$ (harmonic series diverges)

If $p > 1 \Rightarrow p-1 > 0$

$$\Rightarrow \lim_{t\to\infty} \frac{1}{c}\left[\frac{1}{t^{p-1}} - 1\right] = 0$$

Theorem 8.11. *p. 632* ***Convergence of the p-Series (The p-Series Test)***

The p-**series** $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges for all $p > 1$ and diverges for all $p \leq 1$.

Example 8.4.3a) Determine if the series

$$\sum_{k=1}^{\infty} k^{-3}$$

converges

Solution: Since $\sum_{k=1}^{\infty} k^{-3} = \sum_{k=1}^{\infty} \frac{1}{k^3}$ is a p-series

with $p = 3$, this series converges by thm 8.11 ✓

Example 8.4.3b) Determine if $\sum_{k=1}^{\infty} \frac{1}{\sqrt[4]{k^3}}$ converges.

Solution: Since $\sum_{k=1}^{\infty} \frac{1}{\sqrt[4]{k^3}} = \sum_{k=1}^{\infty} \frac{1}{k^{3/4}}$ is a p-series

with $p = 3/4$, this series diverges by thm 8.11 ✓

Estimating the Sum of a Series

One of the major ideas behind sequences & series is to replace a function evaluation problem of type

evaluate $F(x)$ at $x = c \in \text{Dom}(F)$

with the problem:

Find $\sum_{n=1}^{\infty} a_n$ for appropriately chosen a_n

In other words, construct $\{a_n\}_{n=1}^{\infty}$ such that

$$\sum_{n=1}^{\infty} a_n \text{ "}=\text{" } F(x)$$ for $x \in \text{Dom}(F)$ chosen appropriately

$$\Rightarrow \lim_{N\to\infty} \sum_{n=1}^{N} a_n = F(x)$$

$\Rightarrow$ Choose N so that $\sum_{n=1}^{N} a_n \approx F(x)$

$\Rightarrow$ Choose N s.t. $\left| F(x) - \sum_{n=1}^{N} a_n \right|$ sufficiently small

To conclude that $F(x)\Big|_{x=c} = S = \sum_{n=1}^{\infty} a_n$,

we study the remainder between the desired limit S and the Nth term in our partial sums

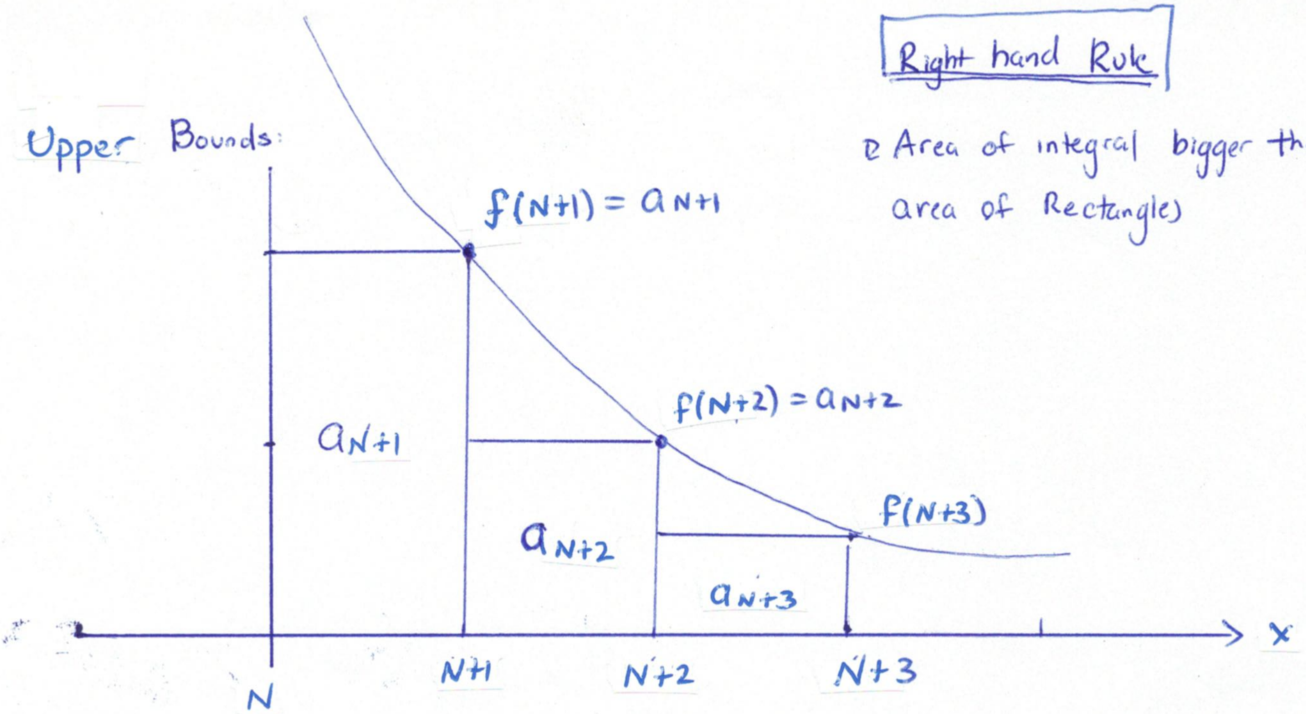
$$R_N = S - S_N$$

$$= \underbrace{\sum_{n=1}^{\infty} a_n}_{\text{infinite series}} - \underbrace{\sum_{n=1}^{N} a_n}_{\text{first } N \text{ terms}}$$

$$= \sum_{n=N+1}^{\infty} a_n \quad \leftarrow \text{the tail of the series starting at the } (N+1)\text{st term}$$

We say $S = \lim_{N\to\infty} S_N \iff \underbrace{\lim_{N\to\infty} R_N = 0}_{\text{we try to force remainder to zero}}$.

Upper Bounds:

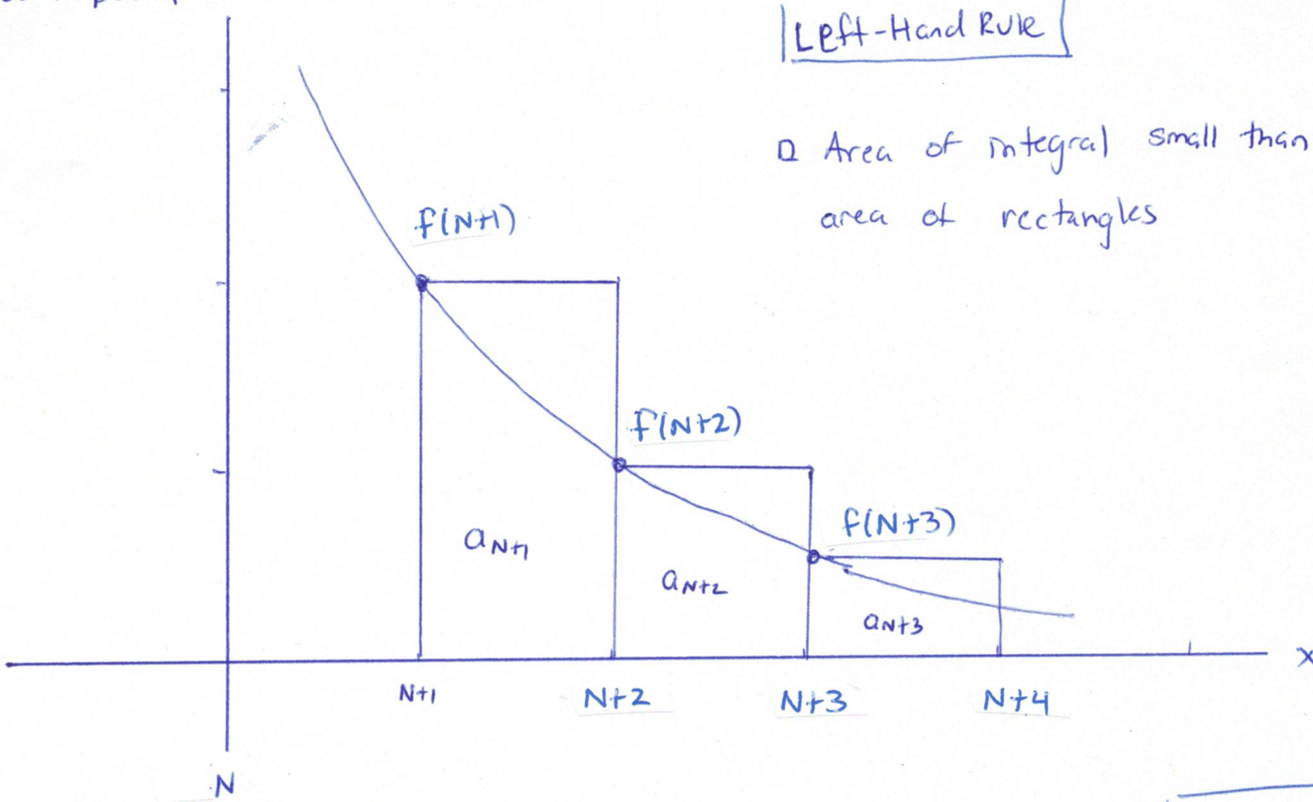


$\boxed{\text{Right hand Rule}}$

(Area of integral bigger than area of Rectangle)

$$R_N = \sum_{n=N+1}^{\infty} a_n = a_{N+1} + a_{N+2} + a_{N+3} + \cdots \leq \int_{n}^{\infty} f(x)\,dx$$

Lower bound



$\boxed{\text{Left-Hand Rule}}$

(Area of integral small than area of rectangles

Theorem 8.12. *p. 635* ***Estimating Series with Positive Terms***

Let $f(x)$ be a continuous, positive decreasing function, for $x \geq 1$, and define sequence $a_k = f(k)$ for all $k \in \mathbb{N}$. Suppose that the limit of the associated convergent infinite series is

$$S = \sum_{k=1}^{\infty} a_k$$

Define the sequence of partial sums

$$S_n = \sum_{k=1}^{n} a_k$$

as the sum of the first n terms of the series. Then, the remainder $R_n = S - S_n$ satisfies the following inequality:

$$R_n < \int_n^{\infty} f(x)dx$$

Furthermore, the exact value of the series satisfies the following bounds:

$$S_n + \int_{n+1}^{\infty} f(x)dx \;<\; \sum_{k=1}^{\infty} a_k \;<\; S_n + \int_n^{\infty} f(x)dx$$

Example 8.4.4 p. 635

How many terms of the convergent p-series

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

must be summed to get within 10^{-3} of the exact value

Solution: Let $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k^2}$.

Since $a_k = \frac{1}{k^2} = f(k)$ with $f(x) = \frac{1}{x^2}$

satisfies the conditions of thm 8.12, we can use our bound

$$R_n = \underset{\text{exact value}}{S} - S_n \underset{\text{by thm 8.12}}{<} \int_n^{\infty} f(x)\,dx < \underbrace{\frac{1}{10^3}}_{\text{desired tolerance}}$$

Ex 8.4.4 p. 635 ...

We need to choose n so that

$$\int_n^\infty \frac{1}{x^2}\,dx < \frac{1}{10^3}$$

$$\Rightarrow \lim_{t\to\infty} \int_n^t x^{-2}\,dx < \frac{1}{10^3}$$

$$\Rightarrow \lim_{t\to\infty} \left[\frac{x^{-1}}{-1}\right]\Bigg|_{x=n}^{t} < \frac{1}{10^3}$$

$$\Rightarrow \lim_{t\to\infty} \left[-\frac{1}{t} + \frac{1}{n}\right] < \frac{1}{10^3}$$

$$\Rightarrow \frac{1}{n} < \frac{1}{10^3}$$

$$\Rightarrow n > 10^3$$

We need to sum at least 1001 terms

Definition. *p. 636* ***Leading Terms and Tail of a Series***

Let N be a positive integer. Given and infinite series

$$\sum_{k=1}^{\infty} a_k$$

the **leading terms** of this series are the terms at the beginning with small index, say all terms a_k with $k < N$. The **tail** of an infinite series consists of the sum of all terms at the "end" of the series with large and increasing index, given by

$$\sum_{k=N}^{\infty} a_k$$

The convergence or divergence behavior of an infinite series depend only on the tail of the series. The value of a convergent series is determined primarily by the leading terms.

Similar to
Example 8.4.4 p. 635

Approx $\sum_{n=1}^{\infty} \frac{1}{n^3}$ using first 10 terms. Estimate the error

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \approx \sum_{n=1}^{10} \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{10^3} \approx 1.1975$$

$$R_{10} = S - S_{10} \le \int_{10}^{\infty} \frac{1}{x^3}\,dx = \lim_{t\to\infty} \left[-\frac{1}{2x^2} \right] \Bigg|_{x=n=10}^{x=t}$$

show for general n

$$= \lim_{t\to\infty} \frac{-1}{2t^2} + \frac{1}{2n^2} \Bigg|_{n=10}$$

$$= \frac{1}{2(10)^2}$$

How many terms required to ensure accuracy $0.0005 = 5 \cdot 10^{-4}$

Need to find N s.t. $R_N \le 0.0005$

$$\int_n^\infty \frac{1}{x^3}\,dx < 0.0005$$

$$\Rightarrow \quad \frac{1}{2n^2} < 0.0005$$

$$\Rightarrow \quad \frac{1}{0.001} < n^2$$

$$\Rightarrow \quad 1000 < n^2 \qquad \sqrt{1000} \approx 31.6$$

$$\Rightarrow \quad 31.6 < n$$

$$\Rightarrow \quad n \ge 32$$

Theorem 8.13. *p. 636* ***Properties of Convergent Series***

Suppose that $c \in \mathbb{R}$ and suppose that

$$\sum_{k=1}^{\infty} a_k = A \qquad \text{and} \qquad \sum_{k=1}^{\infty} b_k = B.$$

Then, as long as we check these conditions, we can conclude

1. *Constant Multiple Law:* $$\sum_{k=1}^{\infty} c \cdot a_k = c \cdot \sum_{k=1}^{\infty} a_k = c \cdot A$$

2. *Sum Law:* $$\sum_{k=1}^{\infty} (a_n \pm b_n) = \sum_{k=1}^{\infty} a_n \pm \sum_{k=1}^{\infty} b_n = A \pm B$$

3. *Fat Tail Law:* If M is a positive integer, then

$$\sum_{k=1}^{\infty} a_k \text{ and } \sum_{k=M}^{\infty} a_k$$

either both converge or both diverge. In general, whether a series converges does not depend on a finite number of terms added to or removed from the series. However, the *value* of a convergent series does change if nonzero terms are added or removed.

Similar to

Example 8.4.5 p. 637

Find the sum of the series

$$\sum_{n=1}^{\infty} \left[\frac{3}{n(n+1)} + \frac{1}{2^n} \right]$$

Solution: Let $a_n = \dfrac{1}{n(n+1)}$ and $b_n = \dfrac{1}{2^n}$

Consider $$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left[\frac{1}{n} - \frac{1}{n+1} \right]$$

$$= \lim_{N \to \infty} \left[\sum_{n=1}^{N} \frac{1}{n} - \frac{1}{n+1} \right].$$

$$= \lim_{N \to \infty} 1 - \frac{1}{N+1}$$

$$= 1$$ (see example 8.2.6)

Also, we know

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \frac{1}{2} \cdot \frac{1}{2^{n-1}}$$

$$= \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{2} \cdot \frac{1}{2^{n-1}}$$

$$= \lim_{N\to\infty} \frac{1}{2} \sum_{n=1}^{N} \left[\frac{1}{2}\right]^{n-1}$$

$$= \lim_{N\to\infty} \frac{1}{2} \cdot \frac{1-\left[\frac{1}{2}\right]^{N+1}}{1-1/2}$$

$$= \frac{1}{2} \cdot \frac{1}{1-1/2} \qquad \text{(Geometric series)}$$

$$= \frac{1}{2} \cdot \frac{1}{1/2}$$

$$= 1$$

Thus, we see

$$\sum_{n=1}^{\infty} \frac{3}{n(n+1)} + \frac{1}{2^n} = 3 \cdot 1 + 1 = \boxed{4}$$