Math 1C: Calculus III

Lesson 18: Ratio, Root, and Comparison Tests

Reference: Brigg's "Calculus: Early Transcendentals, Second Edition"
Topics: Section 8.5: The Ratio, Root, and Comparison Tests, p. 641 - 649

Theorem 8.14. *p. 641* ***Ratio Test***

Let $\sum_{k=1}^{\infty} a_k$ be an infinite series with positive terms $a_k > 0$ for all $k \in \mathbb{N}$. Let

$$r = \lim_{k \to \infty} \frac{a_{k+1}}{a_k}$$

1. If $0 \le r < 1$, then the series converges.
2. If $r > 1$ (including $r = \infty$), then the series diverges.
3. If $r = 1$, then the ratio test is inconclusive.

Note: In words, the ratio test says that the limit of the ratio of successive terms of a positive series must be less than 1 to guarantee convergence of the series.

A major hint to use the ratio test can be found in the format of the sequence terms a_n:

- if the nth term of the series involves $n!$, n^n or r^n for constant $r \in \mathbb{R}$, we might try ratio test.

Idea behind Proof:

If $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = r$, then as n gets bigger,

the ratio $\frac{a_{n+1}}{a_n} \to r \Rightarrow \frac{a_{n+1}}{a_n} \approx r$

$$\Rightarrow \quad a_{n+1} \approx r \cdot a_n$$

Thus, the farther we go out in the series, the terms behave like

$$a_n + a_{n+1} + a_{n+2} + \cdots = a_n + r a_n + r^2 a_n + \cdots$$

$$= a_n \left(1 + r + r^2 + r^3 + \cdots\right)$$

Thus, the tail of this series behaves like a geometric series w/ ratio r. We know the geometric series converges if $0 \le r < 1$ and diverges if $r > 1$.

Again $\rightarrow$ Note:
- a major hint to use ratio test is in the sequence terms a_n.
- If the general nth term of series involves $n!$, n^n, or a^n for constant $a \in \mathbb{R}$, you may want to try the ratio test.

Example 8.5.1a
p. 641

Determine if the following series converges

$$\sum_{n=1}^{\infty} \frac{10^n}{n!}$$

Solution: Let $a_n = \frac{10^n}{n!}$. Let's use the ratio test. To this end, notice $a_n > 0$.

Moreover

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{10^{n+1}}{(n+1)!} \div \frac{10^n}{n!}$$

$$= \lim_{n\to\infty} \frac{10^n \cdot 10}{(n+1)\cdot n!} \cdot \frac{n!}{10^n}$$

$$= \lim_{n\to\infty} \frac{10}{(n+1)} \cdot \frac{10^n}{10^n} \cdot \frac{n!}{n!}$$

$$= \lim_{n\to\infty} \frac{10}{(n+1)}$$

$$= 0 = r < 1$$

By ratio test, the series converges!

Example 8.5.1b Determine if the following series converges
p. 641

$$\sum_{n=1}^{\infty} \frac{n^n}{n!}$$

Solution: Let $a_n = \dfrac{n^n}{n!}$. Let's use the ratio test.

To this end, notice that $a_n > 0$.

Moreover

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} \div \frac{n^n}{n!}$$

$$= \lim_{n\to\infty} \frac{(n+1)\cdot(n+1)^n}{(n+1)\cdot n!} \cdot \frac{n!}{n^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)^n}{n^n} \cdot \frac{n!}{n!} \cdot \frac{(n+1)}{(n+1)}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n$$

$$= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

$$= e = r > 1$$

Thus, this series diverges by ratio test.

Example 8.5.1.c Determine if the following series converge
p. 641

$$\sum_{n=1}^{\infty} e^{-n} \cdot (n^2+4)$$

Solution: Let $a_n = \dfrac{n^2+4}{e^n}$. Let's apply the ratio test in this case. To this end,

notice $a_n > 0$ ✓

Consider $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^2+4}{e^{n+1}} \div \frac{n^2+4}{e^n}$

$$= \lim_{n\to\infty} \frac{n^2+2n+5}{e \cdot e^n} \cdot \frac{e^n}{n^2+4}$$

$$= \lim_{n\to\infty} \frac{1}{e} \cdot \frac{e^n}{e^n} \cdot \frac{n^2+2n+5}{n^2+4}$$

$$= \frac{1}{e} \cdot \left[\lim_{n\to\infty} \frac{n^2+2n+5}{n^2+4} \cdot \frac{1/n^2}{1/n^2}\right]$$

$$= \frac{1}{e} \lim_{n\to\infty} \left[\frac{1+2/n+5/n^2}{1+4/n^2}\right]$$

$= 1/e = r < 1 \Rightarrow$ converges by ratio test. [L18, p

Example: The ratio test is inconclusive for many series. For example, consider

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

Let $a_n = \frac{1}{n} > 0$. Consider

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{n+1} \div \frac{1}{n}$$

$$= \lim_{n\to\infty} \frac{1}{n+1} \cdot \frac{n}{1}$$

$$= \lim_{n\to\infty} \frac{n}{n+1}$$

$$= 1 = r$$

Note: the ratio test is inconclusive yet

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges!

Theorem 8.15. *p. 642* ***Root Test***

Let $\sum_{k=1}^{\infty} a_k$ be an infinite series with nonnegative terms $a_k \geq 0$ for all $k \in \mathbb{N}$. Let

$$\rho = \lim_{k\to\infty} \sqrt[k]{a_k}$$

1. If $0 \leq \rho < 1$, then the series converges.
2. If $\rho > 1$ (including $\rho = \infty$), then the series diverges.
3. If $\rho = 1$, then the root test is inconclusive.

Idea behind proof:

Assume $\rho = \lim_{n\to\infty} \sqrt[n]{a_n}$ exists.

Then for large values of $n \in \mathbb{N}$,

we have $a_n \approx \rho^n$.

Thus, the tail of the series behavies as follows,

$$a_n + a_{n+1} + a_{n+2} + \cdots \approx \rho^n + \rho^{n+1} + \rho^{n+2} + \cdots$$

$$= \rho^n (1 + \rho + \rho^2 + \cdots)$$

The tail of this series behaves like a geometric series w/ ratio ρ. If $0 \le \rho < 1$, the geometric series converges and if $\rho > 1$, the geometric series diverges.

Note: • hints to indicate when to use the root test are found in sequence terms

• If the general nth term of the series involves n as an exponent try the root test.

Example 8.5.2 a
p. 643

Determine if the following series converges

$$\sum_{n=1}^{\infty} \left[\frac{4n^2-3}{7n^2+6}\right]^n$$

Solution: Let $a_n = \left[\frac{4n^2-3}{7n^2+6}\right]^{n} \geq 0$

n ← nth power: think root test

Let's use the root test. To this end, conside

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left[\frac{4n^2-3}{7n^2+6}\right]^n}$$

$$= \lim_{n\to\infty} \frac{4n^2-3}{7n^2+6}$$

$$= \lim_{n\to\infty} \frac{(4n^2-3)}{(7n^2+6)} \cdot \frac{1/n^2}{1/n^2}$$

$$= \lim_{n\to\infty} \frac{4-3/n^2}{7+6/n^2} = \frac{4}{7} = \rho < 1$$

$\Rightarrow$ Series converges via root test.

Example 8.5.2b p.643

Determine if the following series converges

$$\sum_{n=1}^{\infty} \frac{2^n}{n^{10}}$$

nth power: think root test

Solution: Let $a_n = \dfrac{2^{n}}{n^{10}} \geq 0.$

Let's use the root test. Consider

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{2^n}{n^{10}}}$$

$$= \lim_{n\to\infty} \frac{\sqrt[n]{2^n}}{\sqrt[n]{n^{10}}}$$

$$= \lim_{n\to\infty} \frac{2}{(n^{10})^{1/n}}$$

$$= \lim_{n\to\infty} \frac{2}{(n^{1/n})^{10}}$$

note: $\lim_{n\to\infty} n^{1/n} = 1$

$$= \frac{2}{\left[\lim_{n\to\infty} (n^{1/n})\right]^{10}} = 2 = \rho > 1 \Rightarrow \text{diverges by root test.}$$

L18, p.10

Theorem 8.16. *p. 643* ***The (Direct) Comparison Test***

Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be infinite series with positive terms.

1. If $0 < a_k \leq b_k$ for all $k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} b_k$ converge, then the series $\sum_{k=1}^{\infty} a_k$ converges.
2. If $0 < b_k \leq a_k$ for all $k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} b_k$ diverge, then the series $\sum_{k=1}^{\infty} a_k$ diverges.

Note: Whether a series converges depends on the behavior of the terms in the tail of the series. Thus, the inequalities

$$0 < a_k \leq b_k \text{ and } 0 < b_k \leq a_k$$

in this test need not hold for all terms of the series. They must hold for all $k \geq M$ for some positive integer $M \in \mathbb{N}$.

- To use the direct comparison test, we must have some known series $\sum b_n$ for the purpose of comparison.

- Most of the time, we will use
  - a p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$
  - A geometric series $\sum_{n=1}^{\infty} a r^{n-1}$

Testing for Convergence/Divergence by Comparing

Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n + 1}$$

Does this look familiar?

Let $a_n = \dfrac{1}{2^n + 1}$ and set $b_n = \dfrac{1}{2^n}$

Notice $a_n = \dfrac{1}{2^n + 1} < \dfrac{1}{2^n} = b_n$

Further $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{2^n}$

$$= \sum_{n=1}^{\infty} \frac{1}{2} \cdot \left[\frac{1}{2}\right]^{n-1}$$

$$= \frac{1}{2} \cdot \frac{1}{1 - 1/2}$$

$$= \frac{1}{2} \cdot \frac{2}{1}$$

$$= 1 \Rightarrow \sum_{n=1}^{\infty} a_n < 1 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges}$$

Similar to
Example 8.5, 3a p. 644)

Determine whether the series

$$\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$$

converges or diverges.

Solution: Notice that for large values of $n \in \mathbb{N}$, $2n^2$ dominates denom

Further, we know

$$2n^2+4n+3 > 2n^2 \quad \text{for} \quad n \in \mathbb{N}$$

$$\Rightarrow \frac{1}{2n^2+4n+3} < \frac{1}{2n^2}$$

$$\Rightarrow \frac{5}{2n^2+4n+3} < \frac{5}{2n^2}$$

Let $a_n = \dfrac{5}{2n^2+4n+3}$ and $b_n = \dfrac{5}{2n^2}$

Then $a_n \leq b_n \quad \forall n \in \mathbb{N}$.

Further $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

By the comparison test, we know $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

Similar to

Example 8.5.3 b p. 645

Test the series $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ for convergence or divergence.

Solution: Let $a_n = \frac{\ln(n)}{n}$ and $b_n = \frac{1}{n}$.

Notice $\ln(n) > 1 \iff e^{\ln(n)} > e^1$

$\iff n > e$

$\iff n \geq 3$

Thus, if $n \geq 3$, we know $a_n > b_n$

$$\Rightarrow \sum_{n=3}^{\infty} a_n = \sum_{n=3}^{\infty} \frac{\ln(n)}{n} > \sum_{n=3}^{\infty} \frac{1}{n}$$

By our previous discussion, we know

$$\sum_{n=3}^{\infty} \frac{1}{n} \quad \text{diverges}$$

$\Rightarrow \sum_{n=3}^{\infty} a_n$ diverges

$\Rightarrow \sum_{n=1}^{\infty} a_n$ diverges by the comparison test.

Theorem 8.17. *p. 643* ***The Limit Comparison Test***

Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be infinite series with positive terms. Let

$$\lim_{k\to\infty} \frac{a_k}{b_k} = L$$

1. If $0 < L < \infty$ (that is, L is a positive, finite number), then series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ either both converges or both diverge.
2. If $L = 0$ and $\sum_{k=1}^{\infty} b_k$ converge, then the series $\sum_{k=1}^{\infty} a_k$ converges.
3. If $L = \infty$ and $\sum_{k=1}^{\infty} b_k$ diverges, then the series $\sum_{k=1}^{\infty} a_k$ diverges.

Be careful:
- for positive constant limits L in this test, the limit value doesn't directly indicate the convergence behavior (aka: this is different from Ratio/Root test
- the trick is to choose the appropriate b_n sequence

Example: Test the series $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$

for convergence.

Solution: Idea we get is that $\frac{1}{2^n-1}$ looks

similar to $\frac{1}{2^n}$ ← think geometric series ☺

Let $a_n = \frac{1}{2^n-1} > 0$ and $b_n = \frac{1}{2^n} > 0$

Let's try the limit comparison test:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{2^n-1} \div \frac{1}{2^n}$$

$$= \lim_{n\to\infty} \frac{1}{2^n-1} \cdot \frac{2^n}{1}$$

$$= \lim_{n\to\infty} \frac{2^n}{2^n-1} \cdot \frac{1/2^n}{1/2^n}$$

$$= \lim_{n\to\infty} \frac{1}{1-1/2^n} = 1 = L < \infty$$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ and $\sum_{n=1}^{\infty} \frac{1}{2^n}$ have identical

convergence behavior. by limit comparison test.

$$\Rightarrow \text{We Know } \underbrace{\sum_{n=1}^{\infty} \frac{1}{2^n}}_{\text{geometric series}} = \sum_{n=1}^{\infty} \frac{1}{2} \cdot \left[\frac{1}{2}\right]^{n-1}$$

$a = 1/2 \qquad r = \frac{1}{2} < 1$

$\sum_{n=1}^{\infty} a \cdot r^{n-1}$

$$= \frac{a}{1-r} = \frac{1/2}{1-1/2}$$

$\Rightarrow \sum_{n=1}^{\infty} b_n$ converges & thus $\sum_{n=1}^{\infty} a_n$ converges

by limit comparison test.

Example 8.5.4a: Does the series $\sum_{k=1}^{\infty} \frac{5k^4 - 2k^2 + 3}{2k^6 - k + 5}$ converge?

Solution: As $k \to \infty$, we see $\frac{5k^4 - 2k^2 + 3}{2k^6 - k + 5} \approx \frac{5k^4}{2k^6} = \frac{5}{2k^2}$

Thus, why don't we compare w/ p-series $\frac{1}{k^2}$?

$$\text{Let } L = \lim_{k \to \infty} \frac{5k^4 - 2k^2 + 3}{2k^6 - k + 5} \div \frac{1}{k^2}$$

$$= \lim_{k \to \infty} \frac{5k^6 - 2k^5 + 3k^2}{2k^6 - k + 5}$$

$$= \frac{5}{2}$$

Thus, our original series must converge since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges.

Guidelines for Choosing a Test for Series Containing Positive Terms

Here are some reasonable suggestions when testing a series of positive terms for convergence:

1. Begin with the Divergence Test.
2. Ask yourself: "Is the series a special series?" and make sure you can recall the convergence properties of each of the following special series.
   i. **Geometric series**
   ii. $p-$**series**
   iii. **Telescoping series**
   iv. **Harmonic series**
3. If the general kth term of the series look like a function that you can integrate, then try the integral test. Make sure you remember and can apply techniques of integration including:
   - u-substitution
   - Integration by parts
4. If the general kth term of the series involves $k!, k^k$, or a^k for some $a \in \mathbb{R}$, then try the ratio test. Series with k in the exponent may yield to the Root Test.
5. If the general kth term of the series is a rational function of k (or a root of a rational function), use the Direct Comparison Test or the Limit Comparison test with the families of series given in Step 2 above as comparison series.