Math 1C: Calculus III

Lesson 19: Alternating Series

Reference: Brigg's "Calculus: Early Transcendentals, Second Edition"
Topics: Section 8.6: Alternating Series, p. 649 - 660

Definition. *p. 649* ***Alternating Series***

An **alternating series** is a series in the form

$$\sum_{k=1}^{\infty}(-1)^k a_k \qquad \text{or} \qquad \sum_{k=1}^{\infty}(-1)^{k+1} a_k.$$

where $a_k > 0$ for all $k \in \mathbb{N}$. In this case, the signs of each sequence term strictly alternate from positive to negative. The factor $(-1)^k$ or $(-1)^{k+1}$ has the pattern $\{..., 1, -1, 1, -1, 1, -1, ...\}$ and provides the alternating signs on the sequence of terms.

The convergences tests we've considered thus far apply to series w/ positive terms

Alternating Series:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$$

$$\frac{-1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \frac{6}{7} + \cdots = \sum_{n=1}^{\infty} (-1)^{n} \frac{n}{n+1}$$

These two series are examples of a larger class of series known as ALTERNATING SERIES, which have sequence of terms such that

$$(-1)^{n-1} a_n \qquad \text{OR}$$

$$(-1)^{n} a_n$$

where a_n is positive for all $n \in \mathbb{N}$. Here

Theorem 8.18. *p. 650* ***Alternating Series Test***

Let $a_k > 0$ for all $k \in \mathbb{N}$ and consider the alternative series

$$\sum_{k=1}^{\infty}(-1)^{k+1}a_k$$

If we confirm BOTH of the following:

1. The terms of the series are nonincreasing in magnitude ($0 < a_{k+1} \leq a_k$ for k greater than some positive integer M)
2. $\lim_{k\to\infty} a_k = 0$

then the alternating series converges.

Theorem 8.19. *p. 651* ***Alternating Harmonic Series***

The **alternating harmonic series**

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$$

converges (even though the harmonic series diverges).

similar to
Example 8.6.1 : The Alternating Series test is not a silver bullet
p. 652

Consider the series

$$\sum_{n=1}^{\infty} (-1)^n \cdot \frac{3n}{4n-1}.$$

☐ What do we notice about this series?

Let's test for convergence. To this end let $b_n = \dfrac{3n}{4n-1}$

First we try using the alternating series test.

(i) Test for decreasing: Let $f(x) = \dfrac{3x}{4x-1} \Rightarrow f'(x) = \dfrac{(4x-1)\cdot 3 - 4\cdot 3x}{(4x-1)^2}$

$$\Rightarrow f'(x) = \frac{12x - 3 - 12x}{(4x-1)^2} = \boxed{\frac{-3}{(4x-1)^2}}$$

$$\Rightarrow f'(x) < 0 \text{ for } x > 1/4 \checkmark$$

(ii) $\displaystyle\lim_{n\to\infty} \frac{3n}{4n-1} = \lim_{n\to\infty} \frac{3n\cdot\frac{1}{n}}{(4n-1)\cdot\frac{1}{n}}$

$$= \lim_{n\to\infty} \frac{3}{4 - 1/n}$$

$$= 3/4 \neq 0$$

Can't apply alternating series test.

Side Note:

Let $a_n = (-1)^n \cdot \dfrac{3n}{4n-1}$

Then $\displaystyle\lim_{n\to\infty} a_n$ DNE

$\Rightarrow$ By test for divergence we know ~~lim n~~ $\displaystyle\sum_{n=1}^{\infty} a_n$ does not converge.

L19, p6

Similar to

Example 8.6.1a p. 652 :

Test the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$ for convergence / divergence.

Solution: We notice immediately that this is an alternating series

Let $b_n = \frac{n^2}{n^3+1}$. We want to verify our two

conditions from the AST:

(i) $b_{n+1} < b_n$: Let $f(x) = \frac{x^2}{x^3+1}$ where $b_n = f(n)$

$$\Rightarrow f'(x) = \frac{(x^3+1)\cdot 2x - x^2 \overset{(3x^2)}{\cancel{(x^3+1)}}}{(x^3+1)^2}$$

$$\Rightarrow f'(x) = \frac{2x^4 + 2x - 3x^4}{(x^3+1)^2}$$

$$\Rightarrow f'(x) = \frac{2x - x^4}{(x^3+1)^2} = \boxed{\frac{(2-x^3)\cdot x}{(x^3+1)^2}}$$

Since $x > 0$, $f'(x) < 0 \Leftrightarrow 2 - x^3 < 0 \Leftrightarrow x^3 > 2 \Leftrightarrow x > \sqrt[3]{x}$

$\Rightarrow f(n+1) < f(n) \Leftrightarrow n \geq 2$ ✓

(ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^2}{n^3+1} = \lim_{n\to\infty} \frac{n^2}{(n^3+1)} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{n + 1/n^2} = 0$ ✓

Thus, our series converges by the alternating series test. $\boxed{\text{L19, P7}}$

Theorem 8.20. *p. 652* ***Remainder in Alternating Series***

Let $S = \sum_{k=1}^{\infty} (-1)^{k+1} a_k$ be a convergent alternating series with terms that are nonincreasing in magnitude. Let $R_n = S - S_n$ be the remainder in approximating the value of the series by the sum of its first n terms. Then

$$|R_n| \le a_{n+1}$$

In other words, the magnitude of the remainder of a convergent alternating series. is less than or equal to the magnitude of the first neglected term.

L19, p8

Definition. *p. 649* ***Absolute and Conditional Convergence***

If $\sum_{k=1}^{\infty} |a_k|$ converges, then we say $\sum_{k=1}^{\infty} a_k$ **converges absolutely**.

If $\sum_{k=1}^{\infty} |a_k|$ diverges and $\sum_{k=1}^{\infty} a_k$ converges, then we say $\sum_{k=1}^{\infty} a_k$ **converges conditionally**.

Notice $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is conditionally convergent.

Definition: A series $\sum_{n=1}^{\infty} a_n$ is ABSOLUTELY CONVERGENT

if the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ is convergent.

(if $a_n \geq 0 \Rightarrow \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} |a_n|$ & abs. convergence is convergence)

Example 8.4.5 p. 588

Determine if the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ is absolutely convergent

Solution: Let $a_n = \frac{(-1)^{n-1}}{n^2} \Rightarrow |a_n| = \left|\frac{(-1)^{n-1}}{n^2}\right| = \frac{1}{n^2}$

$$\Rightarrow \sum_{n=1}^{\infty} |a_n| = \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}_{\text{convergent using p-series test } (p=2)} = \frac{\pi^2}{6} \checkmark$$

Theorem 8.4.1 p. 588 If a series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then it is convergent.

Idea: $0 \leq a_n + |a_n| \leq 2|a_n| \quad \forall n \in \mathbb{N}$

$\Rightarrow$ If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} 2|a_n|$ converges

$\Rightarrow$ by comparison test $\sum_{n=1}^{\infty} a_n + |a_n|$ converges

$\Rightarrow \sum_{n=1}^{\infty} a_n + |a_n| - \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} a_n \checkmark$

Theorem 8.21. *p. 651* ***Absolute Convergence Implies Convergence***

If $\sum_{k=1}^{\infty} |a_k|$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.

Equivalently, if $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} |a_k|$ diverges.

Note: In words, we say that absolute converges implies convergence.

L19, p.12

Example 8.4.8 p. 590: Using the Ratio Test

Test the series $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{n^3}{3^n}$ for absolute convergence.

Solution: Let $a_n = (-1)^n \cdot \frac{n^3}{3^n}$.

We will use the ratio test for absolute convergence.

Consider
$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\dfrac{(-1)^{n+1} \cdot (n+1)^3}{3^{n+1}}}{\dfrac{(-1)^n \cdot n^3}{3^n}} \right|$$

$$= \left| \frac{(-1)^{n+1}}{(-1)^n} \cdot \frac{(n+1)^3}{n^3} \cdot \frac{3^n}{3^{n+1}} \right|$$

$$= \left| -1 \cdot \left(\frac{n+1}{n}\right)^3 \cdot \frac{1}{3} \right|$$

$$= |-1| \cdot \left| \left(1 + \frac{1}{n}\right)^3 \right| \cdot \left| \frac{1}{3} \right|$$

$$= \left(1 + \frac{1}{n}\right)^3 \cdot \frac{1}{3}$$

We see $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{3}$ and we conclude via ratio test, $\sum_{n=1}^{\infty} a_n$ is absolutely conv
& thus convergent

The Ratio Test: Useful to determine if a series is absolutely convergent

- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ (or $= \infty$), then the series $\sum_{n=1}^{\infty} a_n$ is divergent

- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the ratio test is inconclusive

(no conclusion can be drawn about convergence or divergence)

Note: Series that involve factorials or other products (including a constant raised to the nth power) are often conveniently tested using the ratio test.

Example 8.4.9 p. 590

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$.

Solution: Let $a_n = \frac{n^n}{n!}$. Notice $a_n > 0$ for all $n \in \mathbb{N} \Rightarrow |a_n| = a_n$.

Consider

$$\frac{a_{n+1}}{a_n} = \frac{\left[\frac{(n+1)^{n+1}}{(n+1)!}\right]}{\left[\frac{n^n}{n!}\right]} = \frac{(n+1)^{n+1}}{n^n} \cdot \frac{n!}{(n+1)!}$$

$$= \frac{(n+1)^{n+1}}{n^n} \cdot \frac{\cancel{n!}}{\cancel{n!}\,(n+1)^1}$$

$$= \frac{(n+1)^n}{n^n}$$

$$= \left(\frac{n+1}{n}\right)^n$$

$$= \left(1 + \frac{1}{n}\right)^n$$

$$\Rightarrow \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e > 1$$

$\Rightarrow \sum_{n=1}^{\infty} \frac{n^n}{n!}$ is divergent by Ratio test.

Example of Inconclusive Nature of Ratio test:

Let $a_n = \frac{1}{n^2}$. Let's try to use the ratio test to try to find the convergence behavior of $\sum_{n=1}^{\infty} \frac{1}{n^2}$. To this end, consider

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{n^2}{(n+1)^2} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^2} = 1 \; : \quad \text{We Know } \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ convergent.}$$

On the other hand, consider the divergent series $\sum_{n=1}^{\infty} \frac{1}{n}$. Let $a_n = \frac{1}{n}$.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{n}{(n+1)} \cdot \frac{\frac{1}{n}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1 \; : \quad \text{We Know } \sum_{n=1}^{\infty} \frac{1}{n} \text{ divergent}$$

$\therefore$ If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, we can make no conclusion about $\sum_{n=1}^{\infty} a_n$

Table 8.4 Special Series and Convergence Tests

Series or test	Form of series	Condition for convergence	Condition for divergence	Comments
Geometric series	$\sum_{k=0}^{\infty} a r^k,\ a \neq 0$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	If $\lvert r\rvert < 1$, then $\sum_{k=0}^{\infty} a r^k = \frac{a}{1-r}$.
Divergence Test	$\sum_{k=1}^{\infty} a_k$	Does not apply	$\lim_{k\to\infty} a_k \neq 0$	Cannot be used to prove convergence
Integral Test	$\sum_{k=1}^{\infty} a_k$, where $a_k = f(k)$ and f is continuous, positive, and decreasing	$\int_1^{\infty} f(x)\,dx$ converges.	$\int_1^{\infty} f(x)\,dx$ diverges.	The value of the integral is not the value of the series.
p-series	$\sum_{k=1}^{\infty} \frac{1}{k^p}$	$p > 1$	$p \leq 1$	Useful for comparison tests
Ratio Test	$\sum_{k=1}^{\infty} a_k$, where $a_k > 0$	$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} < 1$	$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} > 1$	Inconclusive if $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = 1$
Root Test	$\sum_{k=1}^{\infty} a_k$, where $a_k \geq 0$	$\lim_{k\to\infty} \sqrt[k]{a_k} < 1$	$\lim_{k\to\infty} \sqrt[k]{a_k} > 1$	Inconclusive if $\lim_{k\to\infty} \sqrt[k]{a_k} = 1$
Comparison Test	$\sum_{k=1}^{\infty} a_k$, where $a_k > 0$	$0 < a_k \leq b_k$ and $\sum_{k=1}^{\infty} b_k$ converges	$0 < b_k \leq a_k$ and $\sum_{k=1}^{\infty} b_k$ diverges	$\sum_{k=1}^{\infty} a_k$ is given; you supply $\sum_{k=1}^{\infty} b_k$.
Limit Comparison Test	$\sum_{k=1}^{\infty} a_k$, where $a_k > 0,\ b_k > 0$	$0 \leq \lim_{k\to\infty} \frac{a_k}{b_k} < \infty$ and $\sum_{k=1}^{\infty} b_k$ converges.	$\lim_{k\to\infty} \frac{a_k}{b_k} > 0$ and $\sum_{k=1}^{\infty} b_k$ diverges.	$\sum_{k=1}^{\infty} a_k$ is given; you supply $\sum_{k=1}^{\infty} b_k$.
Alternating Series Test	$\sum_{k=1}^{\infty} (-1)^k a_k$, where $a_k > 0,\ 0 < a_{k+1} \leq a_k$	$\lim_{k\to\infty} a_k = 0$	$\lim_{k\to\infty} a_k \neq 0$	Remainder R_n satisfies $\lvert R_n\rvert \leq a_{n+1}$
Absolute Convergence	$\sum_{k=1}^{\infty} a_k,\ a_k$ arbitrary	$\sum_{k=1}^{\infty} \lvert a_k\rvert$		Applies to arbitrary series

L19, p.17