Lesson 1: Vectors in $\mathbb{R}^2$

- [ ] Study Skills HW 1 due at start of next class
- [ ] Warm Up Materials
- [ ] Lesson 1: Content

Math 1AB Storyline:

Single-variable calculus studies the ordinary derivative operator on single-variable functions

"Forward" Problem: (Math 1A: ordinary Derivatives)

ordinary derivative operator

$$\frac{d}{dx}\left[F(x)\right] = f(x)$$

Given single variable function

unknown & desired ordinary derivative function

"Backward" Problem (Math 1B: ordinary integrals)

$$\frac{d}{dx}\left[F(x)\right] = f(x)$$

unknown & desired antiderivative function

Given single-variable derivative function

Notice, in both cases, the functions $F(x)$ and $f(x)$ have real-valued output and a single input variable x.

Ordinary derivatives only make sense for single-variable functions

Math 1CD story line:

Multivariable calculus generalizes the ordinary derivative. In particular, in part 1 of math 1C we will study partial derivatives of multivariable functions (functions with more than one input variable).

"Forward" Problem

Math 1C: Find the gradient via partial derivatives

Gradient operator

Unknown and desired partial derivative function with respect to x

$$\vec{\nabla}\left[F(x,y)\right] = \langle f_x(x,y),\ f_y(x,y)\rangle$$

Given multivariable function (In this case we have 2 input variables)

unknown and desired partial derivative function with respect to variable y

"Backward" Problem:

Math 1D: Find the "inverse" of the gradient operation

$$\vec{\nabla}\left[F(x,y)\right] = \underbrace{\langle f_x(x,y),\ f_y(x,y)\rangle}_{}$$

Unknown and desired multivariable function

Given vector field each of whose components are partial derivative functions

In order to understand multivariable functions

$$z = F(x, y)$$

we begin by introducing the ideas of vectors.

First, lets look at vectors with <u>two</u> components.

Points in $\mathbb{R}^2$ (2-Dimensional Cartesian Coordinate System)

Recall that any point in $\mathbb{R}^2$ is defined as an ordered pair

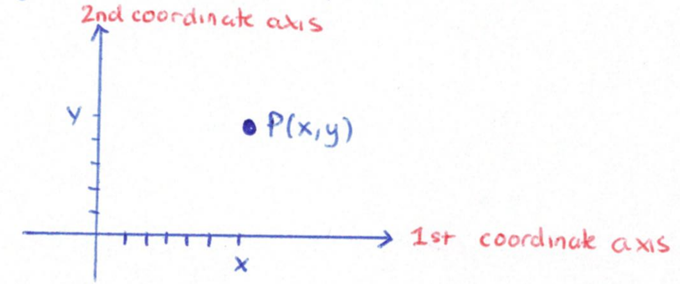
1st coordinate value

$$P(x, y)$$

point name

2nd coordinate value

When we graph this point, we move x-units on the 1st coordinate axis and y-units on the 2nd coordinate axis



Notice, we use parenthesis (round brackests) to denote points in $\mathbb{R}^2$

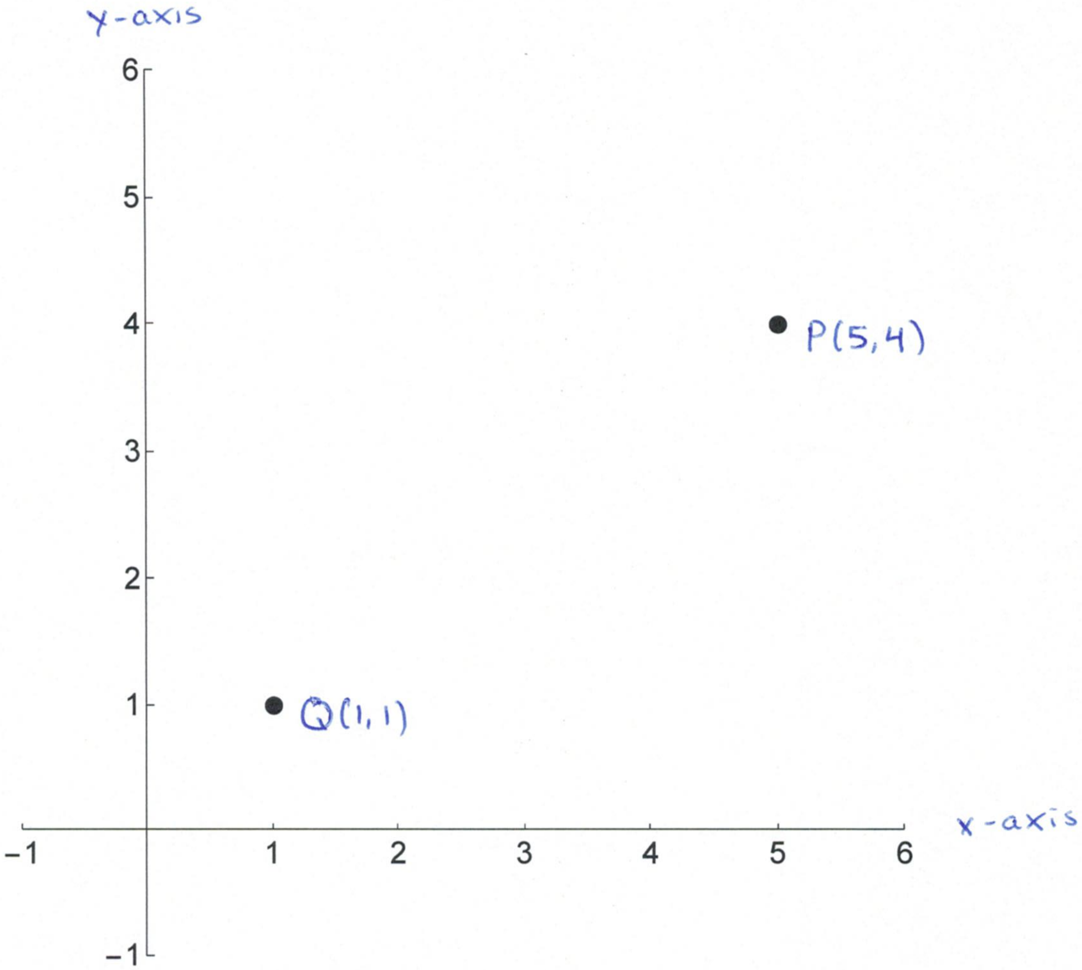
point name ↓ P $\underset{\uparrow \text{ left parenthesis}}{(}$ $\overset{\text{1st coordinate}}{x}$, $\overset{\text{2nd coordinate}}{y}$)

right parenthesis

Example: Let's graph the points

$Q(1, 1)$ and $P(5, 4)$

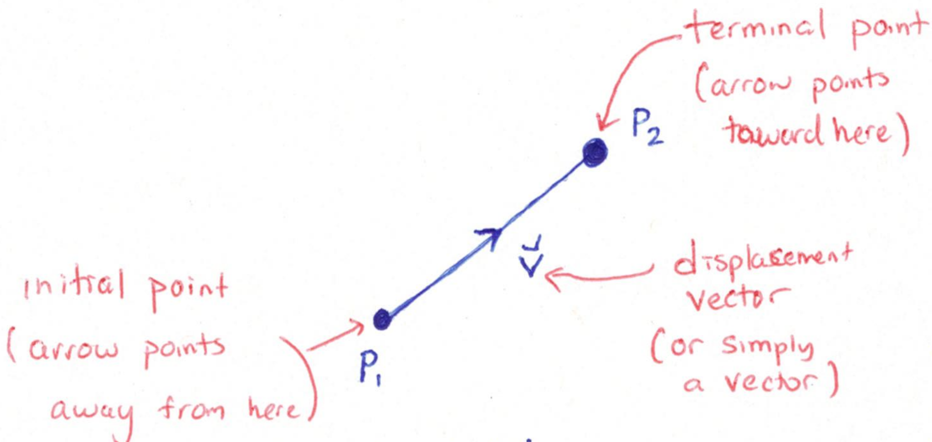
on the xy-axis (see next page).



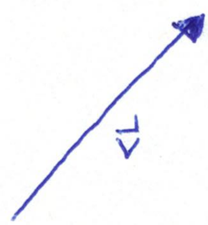
Point P (5,4) represents a "dot" in the xy plane as seen above. Notice: there is no "connectivity" between points, no "direction", and no extra information here.

Classic Calculus Definition: • "a quantity that has both magnitude and direction"

• in Math IC and Math ID, vectors often represented by an arrow or a directed line segment



directed line segment

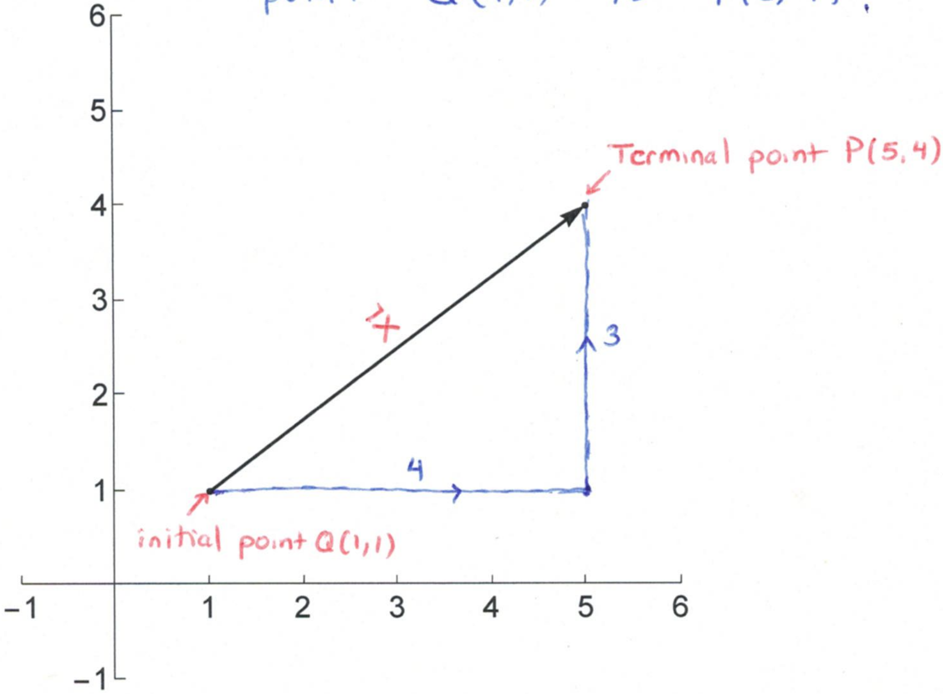


arrow representation

• the "length" of the arrow representation represents the ~~mag~~ magnitude of the vector

relative to origin ⟶ • the "direction" of the vector is ~~the~~ indicated by which way the arrow points. $\boxed{LI, P}$

Example 2: Let's graph vector $\vec{x} = \overrightarrow{QP}$ connecting point $Q(1,1)$ to $P(5,4)$.



We write the vector

1st-component of vector $\vec{x}$

right angle bracket

$$\vec{x} = \langle 4 \ , \ 3 \rangle$$

left angle bracket

2nd component of vector $\vec{x}$

WARNING: There are infinitely many vectors $\vec{x} = \langle 4, 3$
In order to specify $\vec{x}$ uniquely, we need
components and either head or tail point.

A <u>position</u> vector is a vector $\vec{x} = \langle x, y \rangle$ whose tail is the origin. Unless otherwise stated, we assume all vectors are position vectors.

Note about Book's notation:

- Your book will use notation

$$\vec{x} = \langle x_1 , x_2 \rangle = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

left angle braket

right angle bracket

for an orderd pair that refers to a vector.

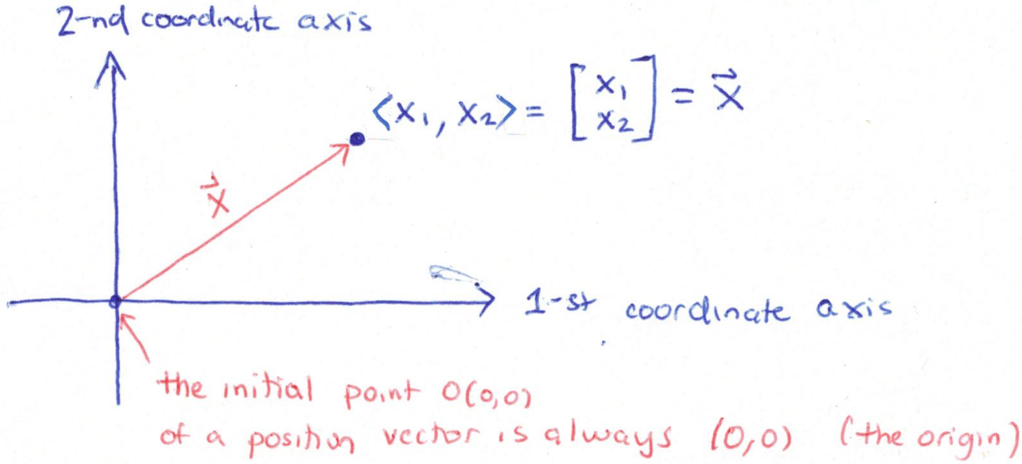
Briggs claims this will help us not confuse vectors with ordered-pair notation (x_1, x_2) representing a point in the plane.

- I think this distinction is very subtle, but I leave it to you to make your own judgement

Components & dimensions

To determine the components of a position vector, we need to choose a coordinate system and treat vector algebraically

$\mathbb{R}^2$ visual



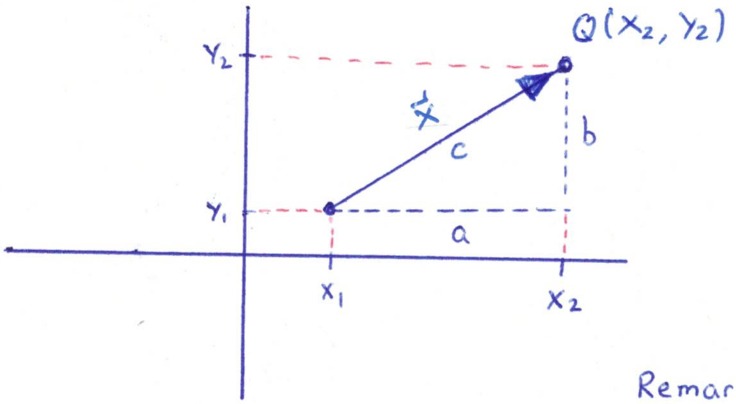
The entries of $\vec{x}$ are called components

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \langle x_1 , x_2 \rangle$$

x_1 : 1st component of vector $\vec{x}$

x_2 : 2nd component of vector $\vec{x}$

Distance formula in $\mathbb{R}^2$:



Remark: $a = x_2 - x_1$

$b = y_2 - y_1$

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ and $\vec{x} = \overrightarrow{PQ}$.
To find the magnitude of $\vec{x}$, we recall the pythagorean theorem

$$c^2 = a^2 + b^2$$

$$= (x_2 - x_1)^2 + (y_2 - y_1)^2 \quad \text{by remark above}$$

$$\Rightarrow c = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \|\vec{x}\|_2 = \|\overrightarrow{PQ}\|_2$$

the distance between P and Q

2-norm of $\vec{x}$

The Length of Position Vector in $\mathbb{R}^2$

For $\vec{x} \in \mathbb{R}^2$ w/ $\vec{x} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \langle x_1, y_1 \rangle$

define the (euclidean) length of the vector as

double vertical bars

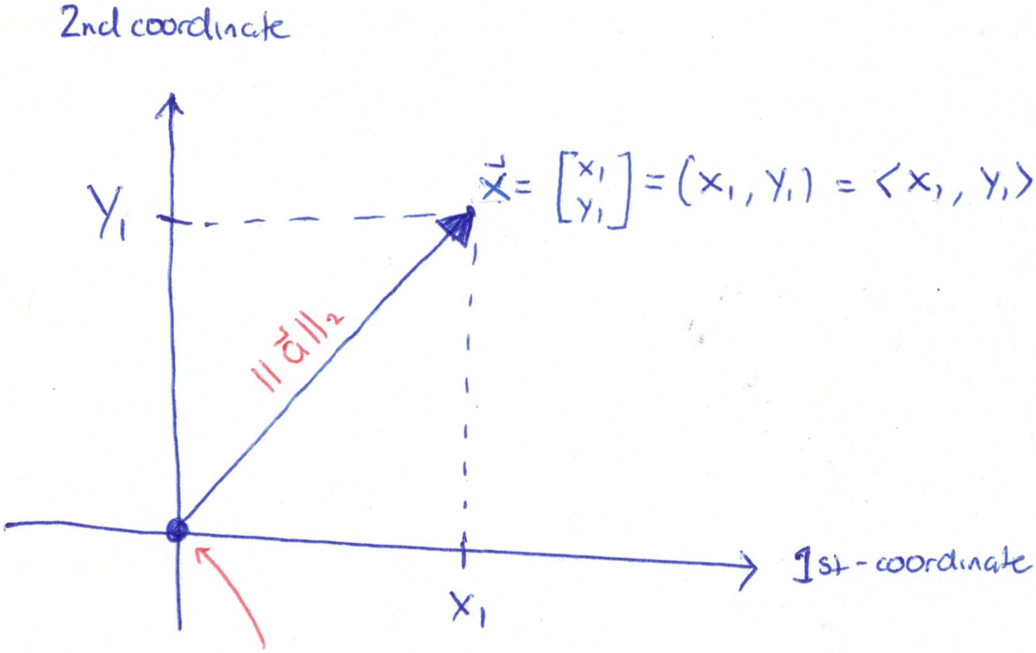
think of the pythagorean thm

$$\|\vec{x}\|_2 = \sqrt{x_1^2 + y_1^2}$$

double vertical bars

subcript 2 on right-hand vertical bars

- The 2-norm of $\vec{a}$ is the length of the vector $\vec{a}$

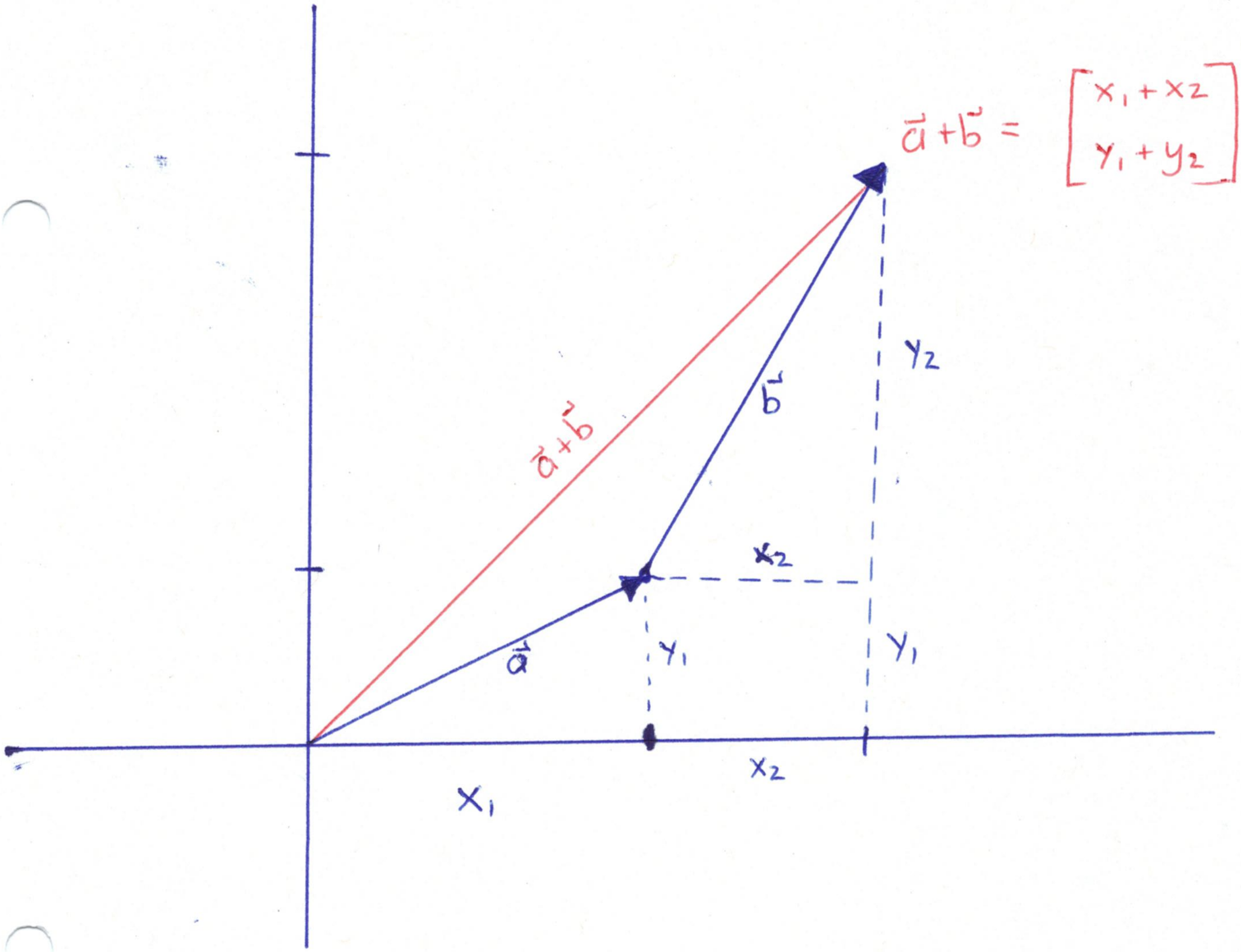


recall: position vectors have tail at origin.

Visualize Vector-add Algebra

Let $\vec{a} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$.

Then $\vec{a} + \vec{b} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}$ is given by



Definition of Vector Add: ■ Let $\vec{u}, \vec{v} \in \mathbb{R}^2$ where

p. 640

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Then $$\vec{u} + \vec{v} = \underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}}_{\text{This is column vector notation}}$$

This is column vector notation :)

■ We can also use ordered-pair notation

$$\vec{u} + \vec{v} = (u_1, u_2) + (v_1, v_2) = (u_1 + u_2, v_1 + v_2)$$

■ Perhaps we prefer row vector notation

$$\vec{u} + \vec{v} = [u_1 \;\; u_2] + [v_1 \;\; v_2] = [u_1 + u_2, v_1 + v_2]$$

■ Elementry basis vector notation

$$\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; j = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\Rightarrow$ ~~th~~ $\vec{u} = u_1 \vec{i} + u_2 \vec{j}$ $\Rightarrow$ $\vec{u} + \vec{v} = (u_1 + v_1)\vec{i}$

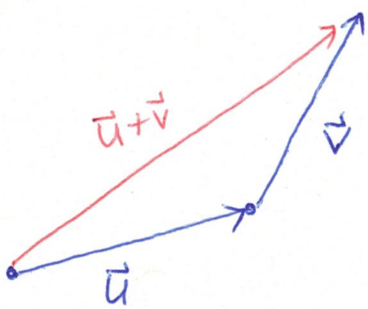
$\vec{v} = v_1 \vec{i} + v_2 \vec{j}$ $\qquad + (u_2 + v_2) j$

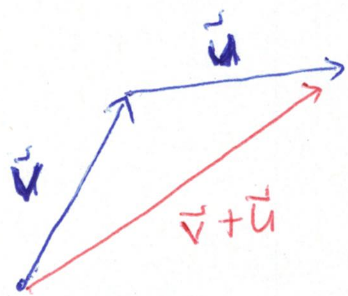
4, p.

Geometric Interpretation of Vector addition p. 640

[Mma Enhanced]

The definition of vector addition is sometimes called the ~~Triangles~~ Triangle Law:





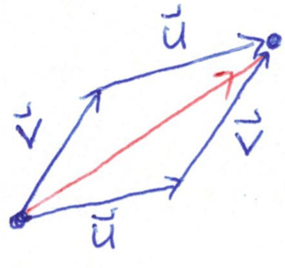
$\vec{u}+\vec{v}$

first $\vec{u}$ then $\vec{v}$

$\vec{v}+\vec{u}$

first $\vec{v}$ then $\vec{u}$

Realizing that $\vec{u}+\vec{v}=\vec{v}+\vec{u}$ can be visualized using the parallelogram Law:



Definition of scalar-vector multiplication:

- Let $c \in \mathbb{R}$ and $\vec{v} \in \mathbb{R}^2$. Then the scalar multiple $c \cdot \vec{v}$ is a vector whose length is $|c|$ times the length of $\vec{v}$ and whose direction is the same as $\vec{v}$ if $c > 0$ and opposite if $c < 0$. (if $c = 0$, then $c \cdot \vec{v} = \vec{0}$)

- In column-vector notation, we ~~ee~~ have

$$c \cdot \vec{v} = c \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix}$$

- Using ordered-pair notation, we have

$$c \cdot \vec{v} = c(v_1, v_2) = (cv_1, cv_2)$$

- For row vector notation, we have

$$c \cdot \vec{v} = c \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} cv_1 & cv_2 \end{bmatrix}.$$

- Elementary basis vector notation requises

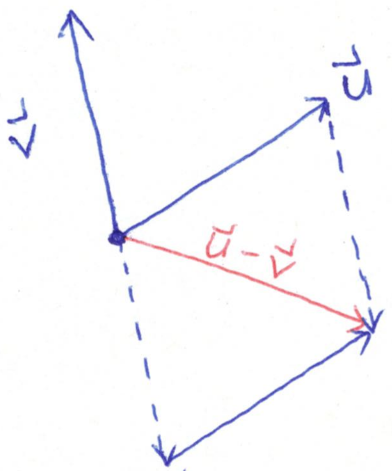
$$c \vec{v} = c(v_1 \vec{i} + v_2 \vec{j})$$
$$= (cv_1)\vec{i} + (cv_2)\vec{j}.$$

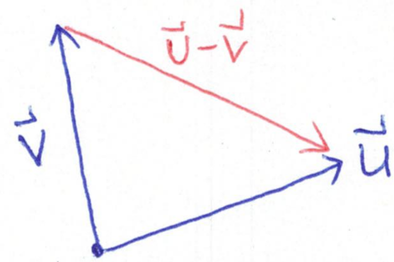
- The vector $-\vec{v} = -1 \cdot \vec{v}$ is called the negative of $\vec{v}$

- By the difference $\vec{u} - \vec{v}$ of two vectors we mean

$$\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$$

- To construct $\vec{u} - \vec{v}$, first draw the negative of $\vec{v}$, then add $-\vec{v}$ to $\vec{u}$ using the parallelogram Law:





Another way to interpret $\vec{u} - \vec{v}$ is to realize

- $(\vec{u} - \vec{v}) + \vec{v} = \vec{u}$
- $\vec{v} + (\vec{u} - \vec{v}) = \vec{u}$

$\vec{u}$ and $\vec{v}$

- We say two vectors are equivalent (or equal) if they have the same length and the same direction. We write $\vec{u} = \vec{v}$

- Another way is to check if the length of

$$\| \vec{u} - \vec{v} \|_2 = 0.$$

- The zero vector, denoted $\vec{0}$, has length zero and no specific direction.