Math 1C: Calculus III

Lesson 21: Properties of Power Series

Reference: Brigg's "Calculus: Early Transcendentals, Second Edition"
Section 9.2: Properties of Power Series, p. 675 - 684

Definition. *p. 676* ***Power Series (Centered at a)***

A **power series** has the general form

$$\sum_{k=0}^{\infty} c_k (x-a)^k,$$

where the scalar $a \in \mathbb{R}$ is a constant real number, the sequence terms c_k are constant and x is variable. The sequence terms $\{c_k\}_{k=0}^{\infty}$ are known as the **coefficients** of the power series and scalar a is called the **center** of the power series. The set of all values of variable x for which the series converges is called the **interval of convergence**, denoted as an interval $I \subseteq \mathbb{R}$. The distance from the center of the interval of convergence to the boundary of the interval is called the **radius of convergence** and is denoted by R.

Remark: When writing out the $k=0$ term, we adopt the convention that

$$(x-a)^0 = 1$$

even when $x=a$. Thus, since all terms

$$c_k \cdot (x-a)^k = 0 \quad \text{for} \quad k \geq 1,$$

the series always converges when $x=a$.

Similar to Example 9.2.1 b p. 676) For what values ~~does~~ of x does the series

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n} \text{ converge?}$$

Solution: Consider the series given in this problem.

Let $a_n = \frac{(x-3)^n}{n}$.

□ What test might we use here?

$$\Rightarrow \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n}\right|$$

$$= \frac{n}{n+1} \cdot |x-3|$$

$$= \frac{1}{1+\frac{1}{n}} \cdot |x-3|$$

$$\Rightarrow \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} \cdot |x-3|$$

$$= |x-3|$$

By the ratio test, we know our series is ~~convergent~~ absolutely convergent (and thus convergent) iff $|x-3|<1$

$$\Rightarrow -1 < x-3 < 1$$

$$\Rightarrow 2 < x < 4$$

- We also know by the ratio test that our series diverges when $|x-3|>1$

$$\Rightarrow \quad x-3>1 \quad \text{or} \quad x-3<-1$$

$$\Rightarrow \quad x>4 \quad \text{or} \quad x<2$$

- we recall that the ratio test is inconclusive (gives no information) when $|x-3|=1$. Then, we consider $x=2$ and $x=4$ separately.

Case $x=4$: $\displaystyle\sum_{n=0}^{\infty} \frac{(x-3)^n}{n} = \sum_{n=0}^{\infty} \frac{(4-3)^n}{n} = \sum_{n=0}^{\infty} \frac{1}{n}$ divergent

Case $x=2$: $\displaystyle\sum_{n=0}^{\infty} \frac{(x-3)^n}{n} = \sum_{n=0}^{\infty} \frac{(2-3)^n}{n}$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n} \quad \text{convergent.}$$

Thus, the given series $\displaystyle\sum_{n=0}^{\infty} \frac{(x-3)^n}{n}$ converges for $2 \le x < 4$.

Example 9.2.1 c) p. 676

A Power Series that converges at only one point

For what $x \in \mathbb{R}$ does the series

$$\sum_{n=0}^{\infty} n!\, x^n$$

converge?

Solution: We see the sequence of terms $a_n = n!\, x^n$ involves both a factorial and a power. Thus, we'll try the ratio test.

To this end consider (when $x \neq 0$)

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!\, x^{n+1}}{n!\, x^n} \right|$$

$$= \lim_{n\to\infty} (n+1)\,|x| \to \infty$$

Then, by the ratio test $\sum_{n=0}^{\infty} n!\, x^n$ is divergent for $x \neq 0$.

On the other hand, if $x = 0$,

$$\sum_{n=1}^{\infty} n!\, x^n = \sum_{n=1}^{\infty} 0 = 0 \quad \text{converges} \checkmark$$

Thus, the given series converges only for $x = 0$. $\square$

Quick Check 9.2.2 p. 678

Consider the power series $\sum_{n=0}^{\infty} x^n$ (where $c_n = 1$).

To find the x-values for which this series converges,

we refer back to our derivation of results for the geometric series

$$\sum_{n=0}^{\infty} x^n = \lim_{N\to\infty} \sum_{n=0}^{N} x^n$$

$$= \lim_{N\to\infty} \frac{1-x^{N+1}}{1-x}$$

We see this limit exists and is finite iff $|x|<1 \Rightarrow -1<x<1$

$$\Rightarrow \text{The power series } \sum_{n=0}^{\infty} x^n \begin{cases} \text{converges if} & -1<x<1 \\ \text{diverges if} & |x| \geq 1 \end{cases}$$

Theorem 9.3. *p. 678* ***Convergence of Power Series***

A power series $\sum_{k=0}^{\infty} c_k (x-a)^k$, centered at a converges in one of three ways:

1. **Infinite Radius of Convergence**

   The series converges for all values of variable $x \in \mathbb{R}$. In this case, the interval of converges is the entire real number line $I = \mathbb{R}$, denoted as interval $(-\infty, \infty)$ and the radius of convergence is $R = \infty$.

2. **Finite, Positive Radius of Convergence**

   There is a real number $R > 0$ such that the series converges for all $|x - a| < R$ and diverges for all $|x - a| > R$. In this case, the radius of convergence is the positive number R.

3. **Zero Radius of Convergence**

   The series converges only at $x = a$ and the radius of convergence is $R = 0$.

Interval of Convergence

Case (3): $R=0 \Rightarrow$ The interval of convergence is a single point a

Case (1): $R=\infty \Rightarrow$ The inverval of convergence is all of $\mathbb{R}$

Case (2): $R>0$ & $R<\infty \Rightarrow$

$$|x-a|<R \Rightarrow -R<x-a<R$$

$$\Rightarrow a-R<x<a+R$$

$\Rightarrow$ Endpoints $x=a\pm R$ are wildcards (anything can happen)

$\Rightarrow$ There are four possibilities for the interval of convergence

$$(a-R,\ a-R)$$

$$(a-R,\ a+R]$$

$$[a-R,\ a+R)$$

$$[a-R,\ a+R]$$

Similar to

Example 9.2.2 p. 678) Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$$

Solution: Let $a_n = \dfrac{(-3)^n \cdot x^n}{\sqrt{n+1}}$

$$\Rightarrow \left|\frac{a_{n+1}}{a_n}\right| = \left| \frac{(-3)^{n+1} \quad x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n \cdot x^n} \right|$$

$$= \left| -3x \cdot \frac{\sqrt{n+1}}{\sqrt{n+2}} \right|$$

$$= 3 \cdot |x| \cdot \frac{\sqrt{1 + 1/n}}{\sqrt{1 + 2/n}}$$

$$\Rightarrow \lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| = 3|x|$$

$\Rightarrow$ Series converges if $3 \cdot |x| < 1$

$\Rightarrow$ $-\frac{1}{3} < x < \frac{1}{3}$

$\Rightarrow$ $R = 1/3$

Now we test end points:

Case $x = 1/3 \Rightarrow \sum_{n=0}^{\infty} \frac{(-3)^n \cdot \left(\frac{1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ converges by Alt Series Test

Case $x = -1/3 \Rightarrow \sum_{n=0}^{\infty} \frac{(-3)^n \cdot \left(-\frac{1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$ diverges by p-series test

Then we see the interval of convergence is

$$\left\{x \in \mathbb{R} : -\frac{1}{3} < x \leq \frac{1}{3}\right\} = \left(-\frac{1}{3}, \frac{1}{3}\right]$$

Similar to

Example 9.2.1 b p. 676)

Find the Radius of Convergence and interval of convergence for series

$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$$

Solution: Let $a_n = \dfrac{n \cdot (x+2)^n}{3^{n+1}}$

$$\Rightarrow \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)\cdot(x+2)^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{n\cdot(x+2)^n}\right|$$

$$= \left|\frac{n+1}{n} \cdot \frac{(x+2)}{3}\right|$$

$$= \left(1+\frac{1}{n}\right) \cdot \frac{|x+2|}{3}$$

$$\Rightarrow \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{|x+2|}{3}$$

$\Rightarrow$ Series converges if $\dfrac{|x+2|}{3} < 1$ by ratio test

$\Rightarrow$ $|x+2| < 3$ $(R=3)$

$\Rightarrow$ $-3 < x+2 < 3$

$\Rightarrow \quad -5 < x < 1$

$\Rightarrow \quad R = 3$ and we want to check endpoints

Case $x = -5$: $\displaystyle\sum_{n=0}^{\infty} \frac{n(-3)^n}{3^{n+1}} = \frac{1}{3}\sum_{n=0}^{\infty} n \cdot (-1)^n$ diverges by test for divergence

Case $x = 1$: $\displaystyle\sum_{n=0}^{\infty} \frac{n(3)^n}{3^{n+1}} = \frac{1}{3}\sum_{n=0}^{\infty} n$ diverges by test for divergence

$\Rightarrow$ Interval of convegence is $(-5, 1)$

Theorem 9.4. *p. 679* ***Combining Power Series***

Suppose the functions $f(x)$ and $g(x)$ can be represented by convergent power series

$$f(x) = \sum_{k=0}^{\infty} c_k\, x^k \qquad \text{and} \qquad g(x) = \sum_{k=0}^{\infty} d_k\, x^k$$

on the interval I.

1. **Sum and Difference Rule**

   The power series $\sum_{k=0}^{\infty} (c_k \pm d_k)\, x^k$ converges to $f(x) \pm g(x)$ on I.

2. **Multiplication by a power function**

   Suppose that $m \in \mathbb{Z}$ with $k + m \geq 0$ for all terms of the power series

   $$x^m \sum_{k=0}^{\infty} c_k\, x^k = \sum_{k=0}^{\infty} c_k\, x^{m+k}$$

   This series converges to $x^m\, f(x)$ for all $x \neq 0$ in I. If $x = 0$, the series converges to

   $$\lim_{x \to 0} x^m\, f(x).$$

3. **Composition**

   If $h(x) = b\, x^m$ for some positive integer $m \in \mathbb{N}$ and a nonzero real number b, then the power series

   $$\sum_{k=0}^{\infty} c_k \left(h(x) \right)^k$$

   converges to the composite function $f(h(x))$ for all x such that $h(x)$ is in I.

L21, p12

Similar to Example 9.2.3 a p. 679) Find the power series representation of $\dfrac{x^3}{x+2}$

Solution: Let $F(x) = \dfrac{x^3}{x+2} = x^3 \cdot \dfrac{1}{2+x}$

this is exactly x^3 times the function we've just analyzed in example 2

$$\Rightarrow x^3 \cdot \frac{1}{2+x} = x^3 \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n \right]$$

→ this series converges and x^3 is constant for any given x in the interval of convergence.

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^3 \cdot x^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^{n+3}$$

$$= \frac{1}{2} x^3 - \frac{1}{4} x^4 + \frac{1}{8} x^5 - \frac{1}{16} x^6 + \cdots$$

$$= \sum_{n=3}^{\infty} \frac{(-1)^{n-1}}{(2^{n-2})} x^n$$

We can guess that the interval of convergence for this series representation will by

$$x \in (-2, 2)$$

Confirm using Ratio test.

Similar to Example 9.2.3 b p. 679

Find a power series representation of $F(x) = \dfrac{1}{2+x}$

Solution: Let $F(x) = \dfrac{1}{2+x}$

To relate this function to ~~or~~ our Known series $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$,

we do some algebra

$$\frac{1}{2+x} = \frac{1}{2\left(1+\frac{x}{2}\right)}$$

$$= \frac{1}{2} \cdot \frac{1}{1-\left(-\frac{x}{2}\right)}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n \, x^n}{2^n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \, x^n}{2^1 \cdot 2^n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^n}{2^{n+1}}$$

This series converges when $\left|-\frac{x}{2}\right| < 1 \Rightarrow |x| < 2$

Radius of convergences: $R = 2$
Interval of convergence: $-2 < x$

Recall:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = F(x) \qquad \text{for} \quad |x| < 1$$

Radius of convergence: $R = 1$

Interval of Convergence: $-1 < x < 1$

Example 9.2.3c p. 679) Find a new power series from an old one

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

Note:

- $(1+x^2)(1-x^2+x^4) = 1-x^2+x^4+x^2-x^4$
- $(1+x^2)\sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-x)^{2n} + \sum_{n=0}^{\infty} x^2 \cdot (-x)^{2n}$

$$= \sum_{n=1}^{\infty}(-1)^n x^{2n} + \sum_{n=0}^{\infty}(-1)^n x^{2n+}$$

$$= \sum_{n=0}^{\infty} (-x^2)^n$$

$$= 1 - x^2 + x^4 - x^6 + \cdots$$

This new series is a geometric series ~~with~~ which converges when

$$|-x^2| < 1 \Rightarrow x^2 < 1$$

$$\Rightarrow |x| < 1$$

Radius of convergence: $R =$

Interval of convergence: $-1 < x < 1$

We could have reapplied ratio test but that extra work is unnecessary since we can find the Radius of convergence by recognizing similarities w/ geometric series

Theorem 9.5. *p. 680* ***Differentiating and Integrating Power Series***

Suppose the power series

$$\sum_{k=0}^{\infty} c_k \, (x-a)^k,$$

converges for all $|x-a| < R$ and defines a function $f(x)$ on that interval.

1. **Differentiation of Power Series**

   The $f(x)$ is differentiable (and thus continuous) for all $|x-a| < R$. Moreover, we can find the derivative $f'(x)$ by differentiating the power series for f term by term

$$f'(x) = \frac{d}{dx}\left[\sum_{k=0}^{\infty} c_k \, (x-a)^k\right]$$

$$= \sum_{k=0}^{\infty} c_k \, \frac{d}{dx}\left[\,(x-a)^k\,\right]$$ ← term-by-term differentiation

$$= \sum_{k=0}^{\infty} k \, c_k \, (x-a)^k.$$

   for $|x-a| < R$.

2. **Integration of Power Series**

   The indefinite integral of f is found by integrating the power series for f, term by term with

$$\int f(x)\,dx = \int \left[\sum_{k=0}^{\infty} c_k \, (x-a)^k\right] dx$$

$$= \sum_{k=0}^{\infty} c_k \int \left[\,(x-a)^k\,\right] dx$$ ← term-by-term integration

$$= \sum_{k=0}^{\infty} c_k \, \frac{(x-a)^{k+1}}{k+1} + c.$$

   for $|x-a| < R$, where c is an arbitrary constant.

Note: Thm 9.5 states the radius convergence remains the same even though the interval of convergence may change. L21, p.16

Example 9.2.4a p. 680)

Let's express $\dfrac{1}{(1-x)^2}$ as a power series by differentiating our equation

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\Rightarrow f'(x) = \frac{d}{dx}\left[\frac{1}{1-x}\right] = \frac{d}{dx}\left[(1-x)^{-1}\right] = -1(1-x)^{-2} \cdot -1 = \frac{1}{(1-x)^2}$$

$$= \frac{d}{dx}\left[\sum_{n=0}^{\infty} x^n\right]$$

$$= \sum_{n=0}^{\infty} \frac{d}{dx}\left[x^n\right]$$

$$= \sum_{n=0}^{\infty} n\, x^{n-1} = \overset{n=0}{0} + \overset{n=1}{1} + \overset{n=2}{2x} + \overset{n=3}{3x^2} + \overset{n=4}{4x^3} + \cdots$$

$$= \sum_{n=1}^{\infty} n\, x^{n-1}$$

$$= \sum_{n=0}^{\infty} (n+1)\, x^n$$

By thm 8.6.2, we know the radius of convergence is $R = 1$.

Similar to Example 9.2.4b p. 681)

Find a power series representation for $\ln(1+x)$ and its radius of convergence.

Notice $\frac{d}{dx}\left[\ln(1+x)\right] = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n$ for $|x|<1$

$$\Rightarrow \ln(1+x) = \int \frac{1}{1+x}\, dx$$

$$= \int \left[\sum_{n=0}^{\infty} (-1)^n \cdot (x)^n\right] dx$$

$$= \sum_{n=0}^{\infty} \int (-1)^n x^n\, dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \int x^n\, dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

$$= \overset{n=0}{x} - \overset{n=1}{\frac{x^2}{2}} + \overset{n=2}{\frac{x^3}{3}} - \overset{n=3}{\frac{x^4}{4}} + \cdots$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$ where $|x|<1$.

Example 9.2.5 a p. 681)

Find a power series for $f(x) = \tan^{-1}(x)$

Solution: Notice

by example 8.6.1

$$f'(x) = \frac{d}{dx}\left[\tan^{-1}(x)\right] = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n$$

$$\Rightarrow \tan^{-1}(x) = \int \frac{1}{1+x^2}\, dx$$

$$= \int \left[\sum_{n=0}^{\infty} (-1)^n \cdot x^{2n}\right] dx$$

$$= \sum_{n=0}^{\infty} \left[\int (-1)^n x^{2n}\, dx\right]$$

$$= \sum_{n=0}^{\infty} (-1)^n \int x^{2n}\, dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

w/ Radius of convergence $R = 1$.

After testing the endpoints individually, we find the interval of convergence is $[-1, 1]$.