Lesson 2: Vectors in $\mathbb{R}^3$

Recall from last time, one of the major focuses of math 1C is to generalize the results from single variable calculus (Math 1A, Math 1B) to functions w/ ~~multidem~~ multidimensional input data:

single variable case

$$f : D \longrightarrow \mathbb{R}, \qquad D \subseteq \mathbb{R}$$

Math 1A $\longrightarrow$ ▫ Study ordinary derivatives (including techniques & applications)

$$\frac{d}{dx}\Big[\underset{\text{given}}{F(x)}\Big] = \underset{\text{unknown}}{f(x)} = F'(x)$$

Math 1B $\longrightarrow$ ▫ Study ordinary antiderivatives (including techniques & applications)

$$\frac{d}{dx}\Big[\underset{\text{unknown}}{F(x)}\Big] = \underset{\text{given}}{f(x)}$$

Multivariable case

$$f : D \longrightarrow \mathbb{R}, \qquad D \subseteq \mathbb{R}^2, \quad D \subseteq \mathbb{R}^3$$

Math 1C $\longrightarrow$ ▫ study partial derivative (including techniques & applications)

$$\frac{\partial}{\partial x_i}\Big[\underset{\text{"given"}}{F(x_1, x_2)}\Big] = \underset{\text{unknown}}{f(x_1, x_2)}$$

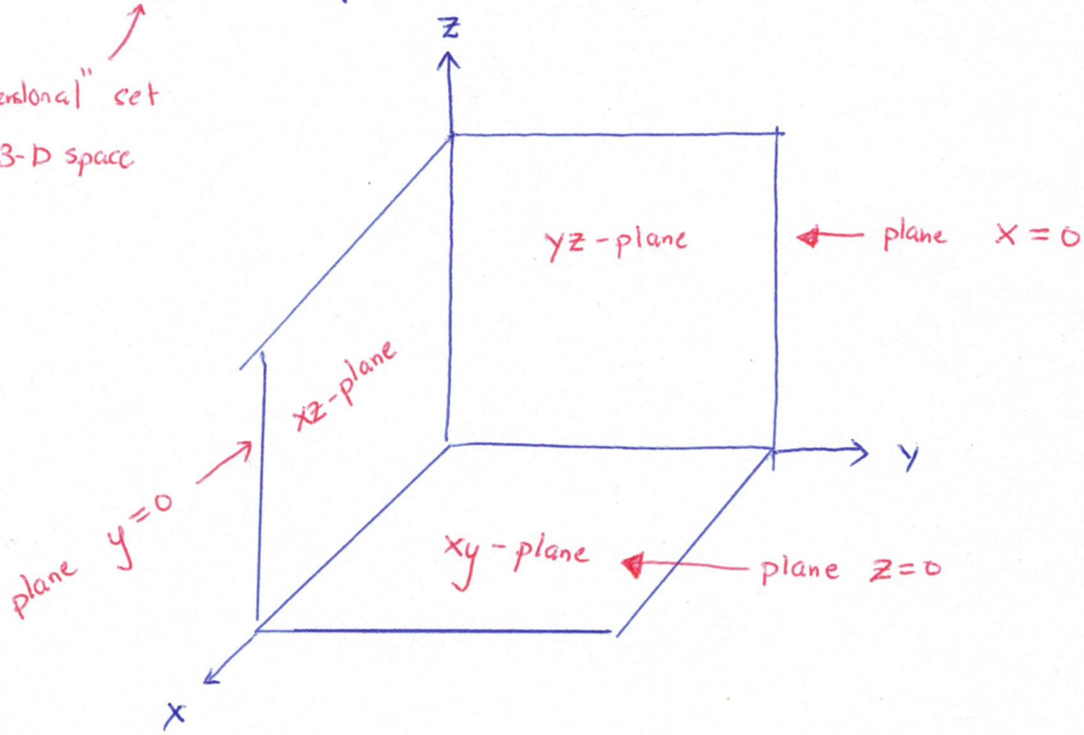
Math 1D $\longrightarrow$ ▫ study partial antiderivatives (including techniques & applications)

$$\frac{\partial}{\partial x_i}\Big[\underset{\text{unknown}}{F(x_1, x_2)}\Big] = \underset{\text{given}}{f(x_1, x_2)}$$

Introduction to $\mathbb{R}^3 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x, y, z \in \mathbb{R} \right\}$

Planes in $\mathbb{R}^3$

"2 Dimensional" set in 3-D space



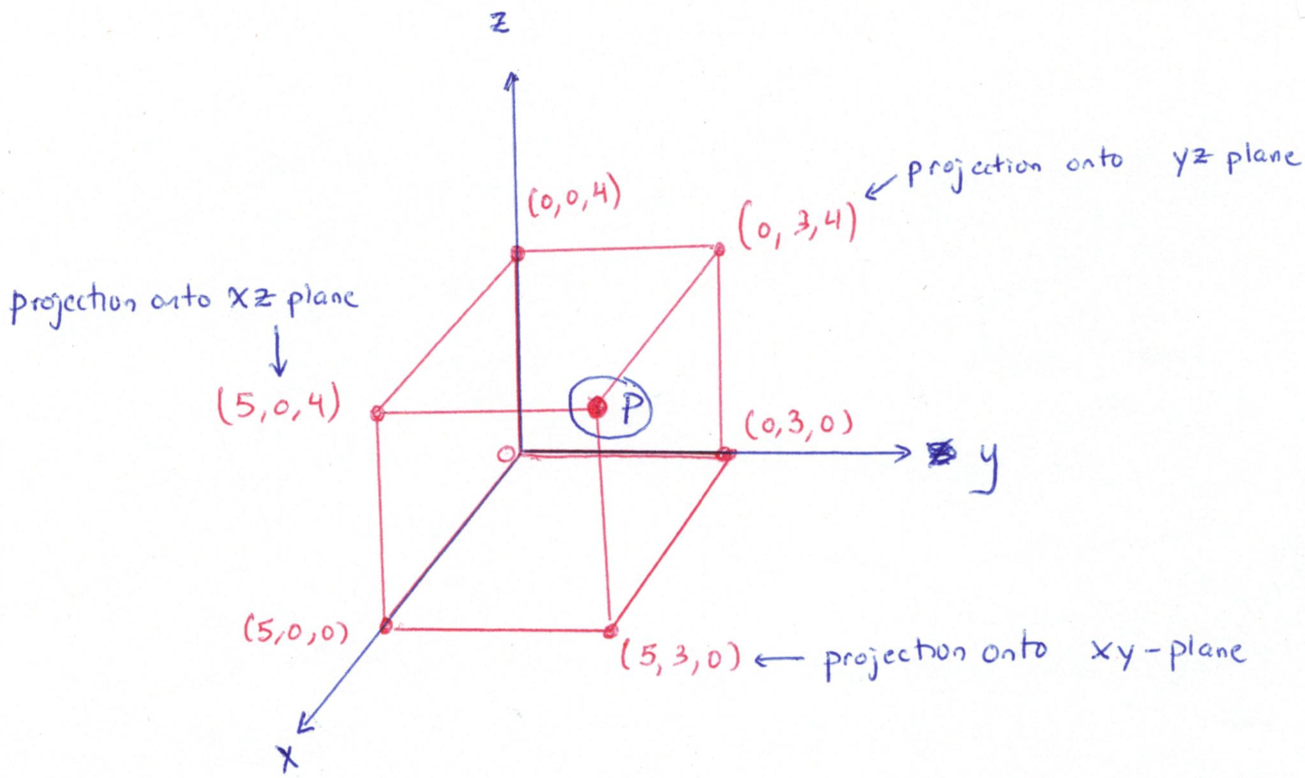
<u>Right hand Rule</u>

- Convention that helps us determine which way we should orient positive z direction

- point fingers in same direction as + x-axis
  curl 90° toward + y-axis
  Thumb points in + z-axis direction

$\mathbb{R}^3$: 3D rectangular coordinate system

Let's graph point $P(5, 3, 4)$



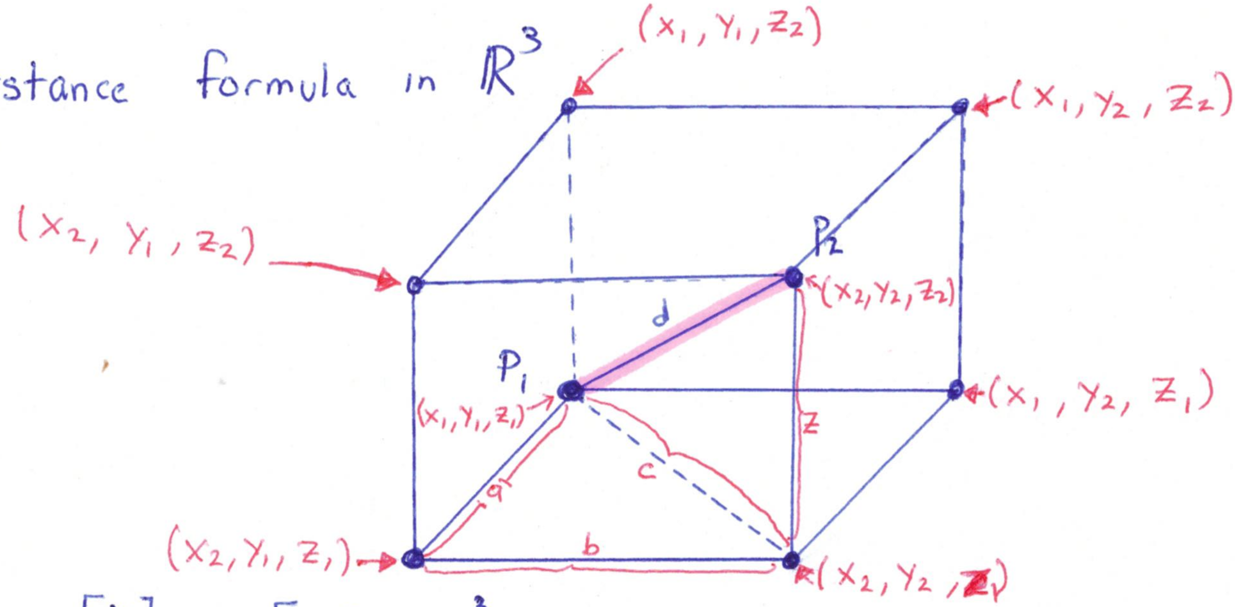
Notice: unit vectors

$$\begin{bmatrix}5\\3\\4\end{bmatrix} = 5\begin{bmatrix}1\\0\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\\0\end{bmatrix} + 4\begin{bmatrix}0\\0\\1\end{bmatrix}$$

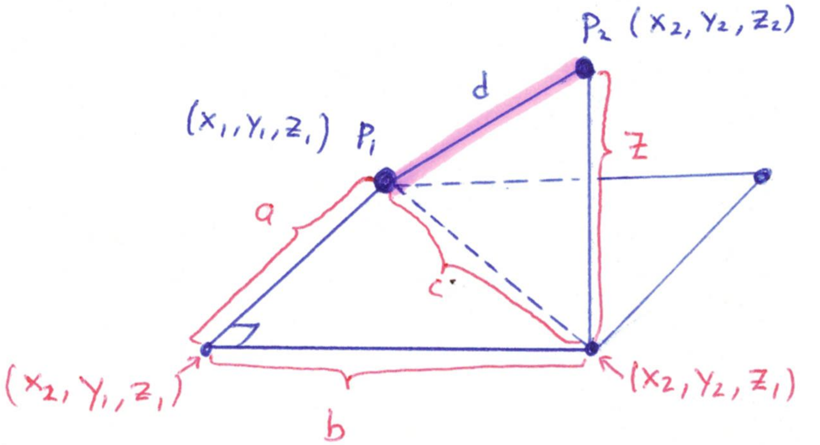
x-direction y-direction z-direction

position vector $\vec{x} = \overrightarrow{OP}$ (with tail at origin $O(0,0,0)$)

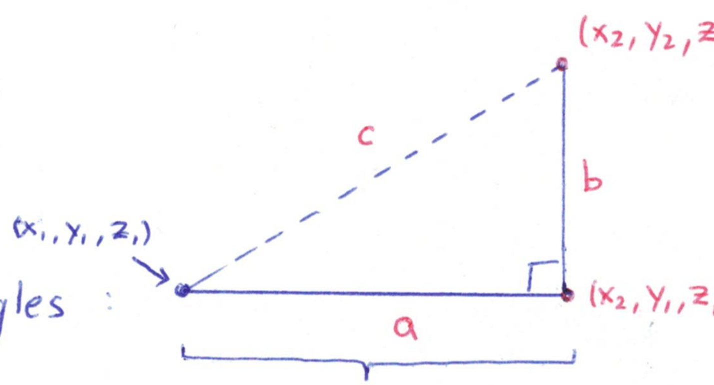
Distance formula in $\mathbb{R}^3$



Let $P_1 \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$, $P_2 \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \in \mathbb{R}^3$. To find the distance between P_1 and P_2 (the length of the line segment highlighted in pink above), consider the simplified diagram



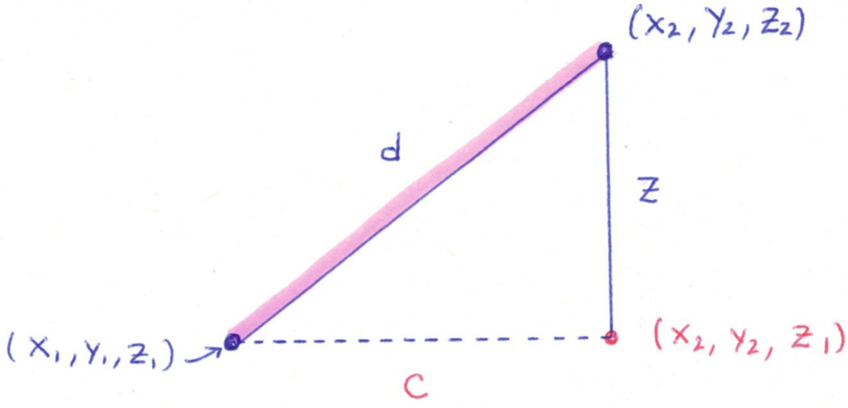
We see two relevent triangles:



Triangle 1 in xy plane

$a = x_2 - x_1$

$b = y_2 - y_1$



Triangle 2 $\qquad z = z_2 - z_1$

By pythagorean theorem, we have

$$d^2 = c^2 + z^2$$

$$= c^2 + (z_2 - z_1)^2 \qquad \text{by above}$$

$$= a^2 + b^2 + (z_2 - z_1)^2 \qquad \text{by note on previous page about triangle 1}$$

$$= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \qquad \text{by equivalent expression for } a, b$$

Thus, the distance between P_1 & P_2 is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Given any points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ the vector $\vec{v}$ with $\vec{PQ}$ is

$$\vec{v} = \langle x_2 - x_1 , y_2 - y_1 , z_2 - z_1 \rangle$$

• vector connecting points P to Q

(start at P, go to Q)

Example 3 p. 642

Find the vector represented by the directed line segment connecting point $P(2, -3, 4)$ to point $Q(-2, 1, 1)$

09/06/2017

Needs Revision

Example: Find distance between $P(2,-1,7)$ and $Q(1,-3,5)$

(or the length of vector $\vec{x} = \overrightarrow{PQ}$

Solution: Let $P(x_1, y_1, z_1) = (2,-1,7)$, $Q(x_2, y_2, z_2) = (1,-3,5)$

~~$P_2(1,-3,5$~~

~~$P_2(1,-3,5)$~~

Then the distance between P and Q is given by

$$\|\vec{x}\| = d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

$$= \sqrt{(1-2)^2 + (-3--1)^2 + (5-7)^2}$$

$$= \sqrt{(-1)^2 + (-2)^2 + (-2)^2}$$

$$= \sqrt{1+4+4} = \sqrt{9} = \boxed{3}. \square$$

The Length of a Vector in $\mathbb{R}^3$

For $\vec{a} \in \mathbb{R}^3$ w/ $\vec{a} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$, we define the euclidean length of our vector as

$$\|\vec{a}\|_2 = \sqrt{x_1^2 + y_1^2 + z_1^2}$$

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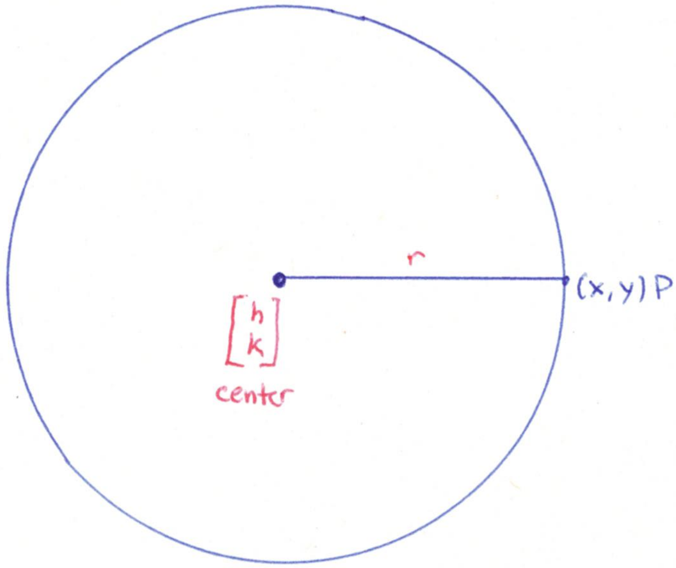
Equations of Spheres in $\mathbb{R}^3$

Let's start by looking at a circle in $\mathbb{R}^2$

This is a shape in $\mathbb{R}^2$ →

(all points have only two coordinates)



- center at (h,k)
- radius r

Recall:

Any point $\begin{bmatrix} x \\ y \end{bmatrix}$ on circle will be distance r away ~~from~~ from Center.

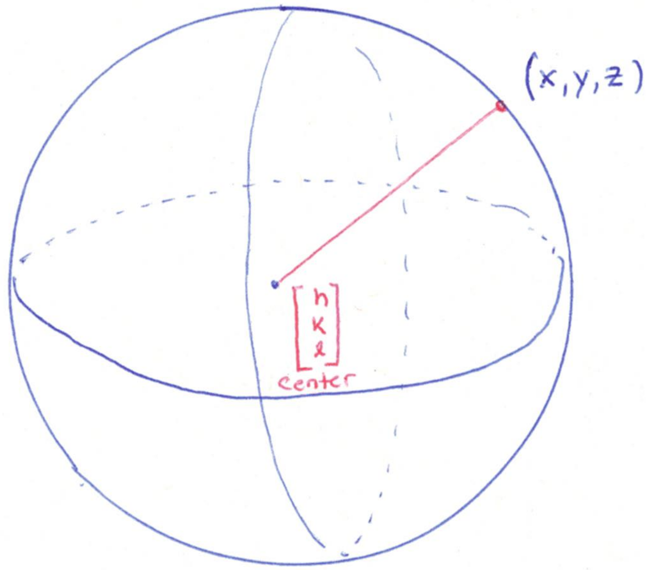
In ~~to~~ other words, for point $P \begin{bmatrix} x \\ y \end{bmatrix}$, the distance between P and $\begin{bmatrix} h \\ k \end{bmatrix}$ is

$$\text{dist}\left(\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} h \\ k \end{bmatrix}\right) = \sqrt{(x-h)^2 + (y-k)^2} = r$$

$$\Rightarrow \quad r = \sqrt{(x-h)^2 + (y-k)^2}$$

The three dimensional analog of this shape would be something like the following

This is a shape in $\mathbb{R}^3$ (all points have exactly three coordinates)



center : (h, K, ℓ)

radius : r

The sphere is the surface for which all points are distance r away from center (h, K, ℓ)

$$\text{dist}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}, \begin{bmatrix} h \\ K \\ \ell \end{bmatrix}\right) = \sqrt{(x-h)^2 + (y-K)^2 + (z-\ell)^2} = r$$

$$\Rightarrow \quad (x-h)^2 + (y-K)^2 + (z-\ell)^2 = r^2$$

$$\Rightarrow \quad (x-h)^2 + (y-K)^2 + (z-\ell)^2 - r^2 = 0$$

$\boxed{\text{Ex 11.2.4 p.774}}$ Finding the equation of a sphere via completing the square

Suppose we have a sphere given by equation

$$x^2 + y^2 + z^2 + 4x - 6y + 2z + 6 = 0$$

To find the radius & coordinates of center, lets complete the square

$$0 = x^2 + y^2 + z^2 + 4x - 6y + 2z + 6$$

$$\Rightarrow 0 = (x^2 + 4x) + (y^2 - 6y) + (z^2 + 2z) + 6$$

we add four to create a perfect square trinomial

$$\Rightarrow 4 + 9 + 1 = (x^2 + 4x + 4) + (y^2 - 6y + 9) + (z^2 + 2z + 1) + 6$$

we must make sure to add constants to both sides of equation

$$\Rightarrow 14 - 6 = (x+2)^2 + (y-3)^2 + (z+1)^2$$

$$\Rightarrow (x - -2)^2 + (y-3)^2 + (z - -1)^2 = (2\sqrt{2})^2$$

center: $\begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}$

radius: $r = 2\sqrt{2}$

$\boxed{\text{L12, p.11}}$

The Equations of Cylinders in $\mathbb{R}^3$

To begin our discussion of cylinders, let us consider the set

two coordinates

radius

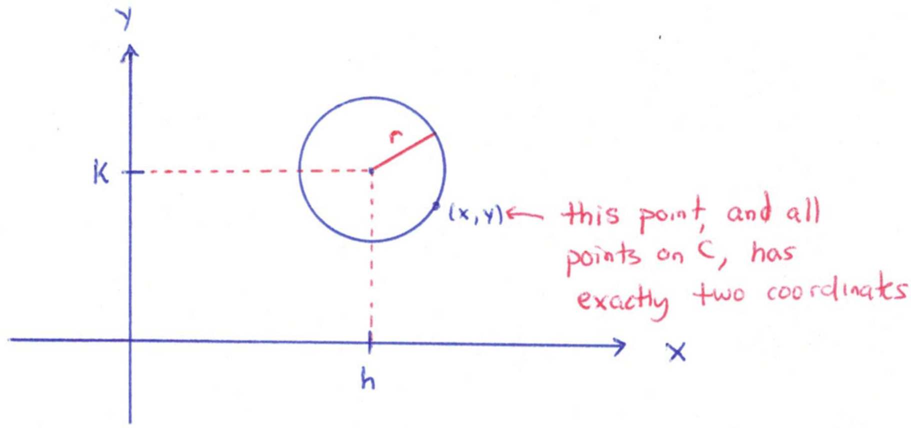
$$C = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : (x-h)^2 + (y-k)^2 = r^2 \right\} \subseteq \mathbb{R}^2$$

x-coordinate of center of circle

y-coordinate of center of circle

where $h, k \in \mathbb{R}$ and $r \in \mathbb{R}$.

Recall, we can visualize this set as a circle centered at (h,k) with radius r.



Remark: Any point (x,y) on the curve C satisfies the relation ~~x^2~~

$$(x-h)^2 + (y-k)^2 = r^2$$

Notice, by definition, all elements of C have <u>two</u> coordinates $\begin{bmatrix} x \\ y \end{bmatrix}$

We can also study curves

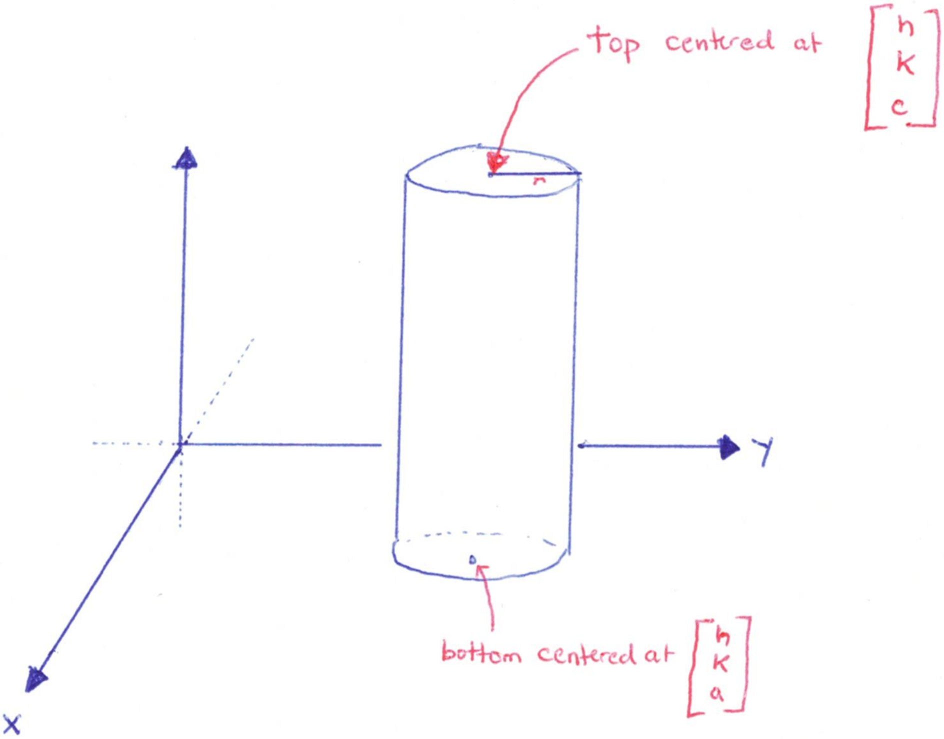
three ~~two~~ coordinates

$$\hat{C} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : (x-h)^2 + (y-k)^2 = r^2, \quad a \leq z \leq c \right\} \subseteq \mathbb{R}^3$$

where $h, k, a, c, r \in \mathbb{R}$.

This set can be visualize this set as a ~~circle~~ cylinder in $\mathbb{R}^3$

centered at (h, k) w/ radius r with height $c-a$



For enhanced example see Mathematica notebook for Class 1

Remark: Pay special attention here. Even though the defining equation for both sets C and $\hat{C}$ are the same ($\cancel{\neq}$ $(x-h)^2 + (y-k)^2 = r^2$), the sets are NOT equal $C \neq \hat{C}$. In particular,

- C sits in $\mathbb{R}^2$: it consists of 2D coordinates $\begin{bmatrix} x \\ y \end{bmatrix}$
- $\hat{C}$ sits in $\mathbb{R}^3$: it consists of 3D coordinates $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

L2, P