Math 1C: Intro to Multivarriable Calculus, Sequences & Series

Lesson 3: Dot Products (Section 11.3)

Announcements:

[I] Show example of "best practices" for completing online homework/quizzes

[a] Print a physical copy

[b] Write down ~~the~~ solutions on seperate sheets

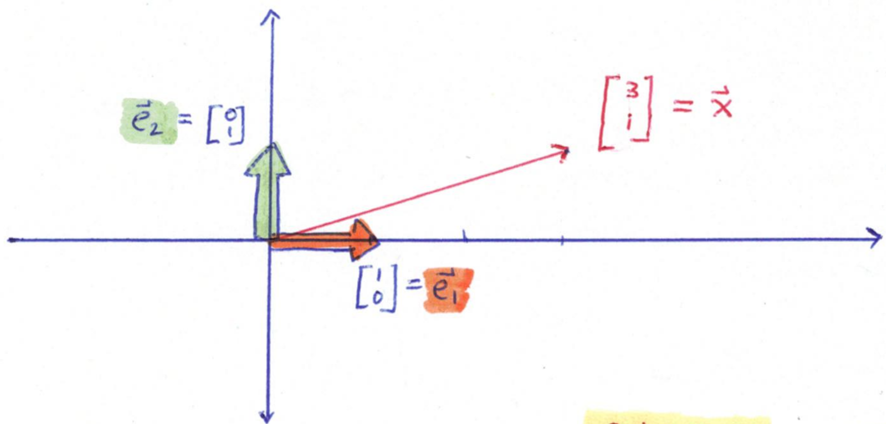
[c] generate questions list

[d] Get ready to stay organized

[II] Study skills 2 Hw due at beginning of class

[III] Last day to drop for full refund: Friday of week 2

Example 1: To begin our discussion, let's look back at vectors in $\mathbb{R}^2$



For vector $\vec{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = (3,1) = \overbrace{3}^{3\cdot 1 + 1\cdot 0} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \overbrace{1}^{3\cdot 0 + 1\cdot 1} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \cancel{3\mathbf{i} + 1\mathbf{k}}$

$$= 3\,\mathbf{i} + 1\,\mathbf{j}$$

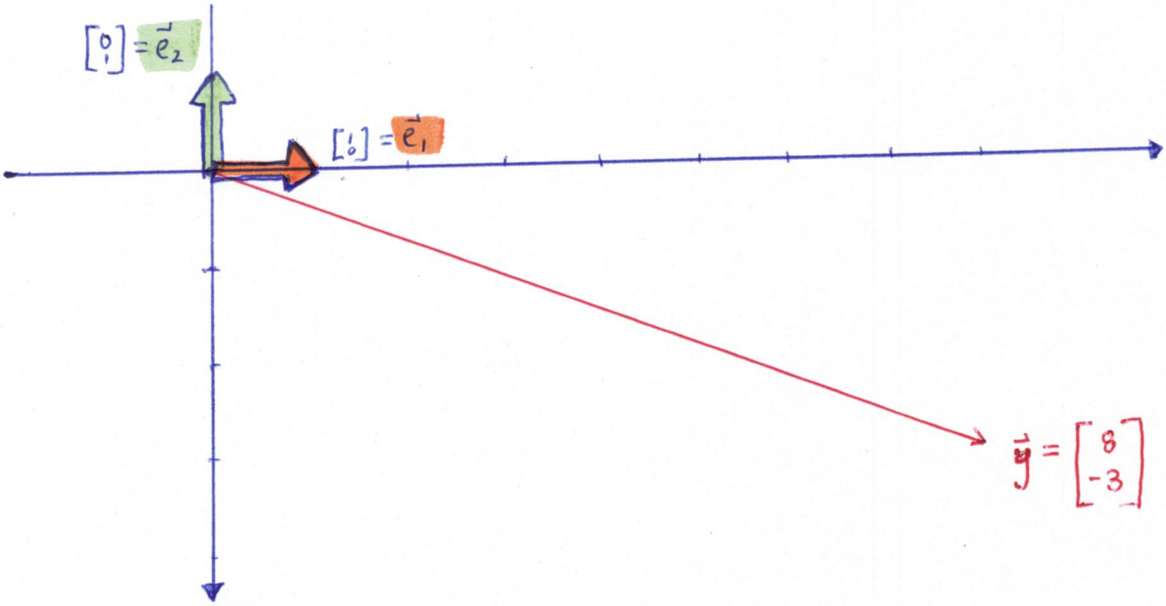
It is often really helpful to be able to discuss how much of

$\vec{x}$ is in the direction of $\vec{e}_1$:

$\vec{x}$ is in the directon of $\vec{e}_2$:

$$\vec{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 \\ 0 \end{bmatrix}}_{\text{proj}_{\vec{e}_1}(\vec{x})} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\text{proj}_{\vec{e}_2}(\vec{x})}$$

Example 2:



- $\text{Proj}_{\vec{e}_1}(\vec{y}) = \begin{bmatrix}8\\0\end{bmatrix} = 8 \cdot \begin{bmatrix}1\\0\end{bmatrix} = 8\,\vec{e}_1 = 8\,\mathbf{i} = \boxed{(8\cdot 1 + -3\cdot 0)} \begin{bmatrix}1\\0\end{bmatrix}$

- $\text{Proj}_{e_2}(\vec{y}) = \begin{bmatrix}0\\-3\end{bmatrix} = -3 \begin{bmatrix}0\\1\end{bmatrix} = -3\,\vec{e}_2 = -3\,\mathbf{j} = \boxed{(8\cdot 0 + -3\cdot 1)} \begin{bmatrix}0\\1\end{bmatrix}$

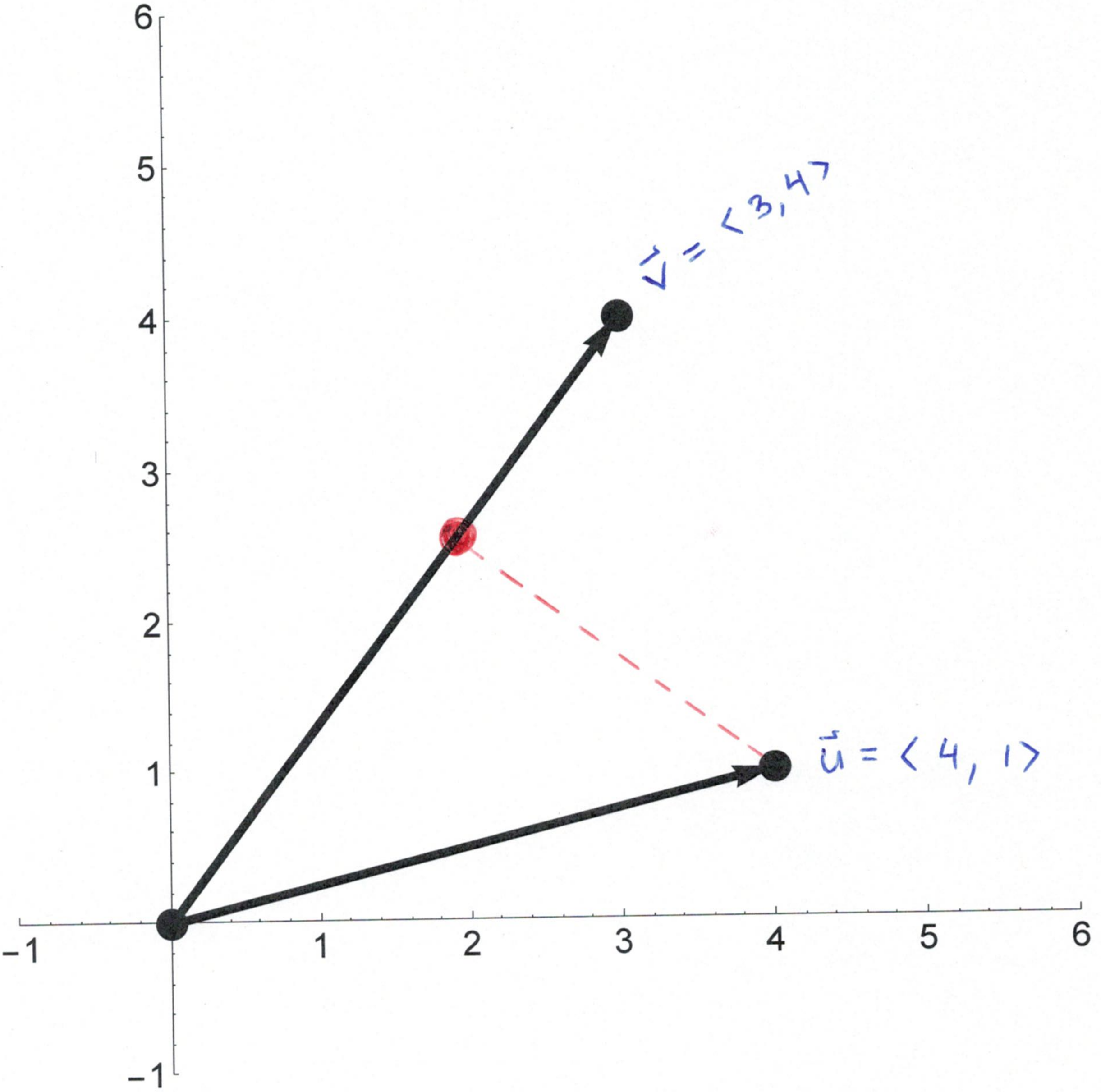
- How can we find the values α, β such that

$$\text{Proj}_{\vec{e}_1}(\vec{x}) = \alpha \begin{bmatrix}1\\0\end{bmatrix}$$

$$\text{Proj}_{\vec{e}_2}(\vec{x}) = \beta \begin{bmatrix}0\\1\end{bmatrix}$$

Notice: In both these examples, the vectors $\vec{e}_1 = \langle 1, 0 \rangle$ and $\vec{e}_2 = \langle 0, 1 \rangle$ are very special.
what if we try a more interesting problem.

Example 11.3.3 a p. 786



Find how much of vector $\vec{u}$ is in the direction of vector $\vec{v}$.

The general question that arises here is

- Given any two vectors $\vec{x}\ \vec{y}$, how do we Know how to find

$$\text{proj}_{\vec{x}}(\vec{y}) \qquad \text{or} \qquad \text{proj}_{\vec{y}}(\vec{x})$$

Enter : new operation called the dot product (inner product)

Dot Product in $\mathbb{R}^2$:

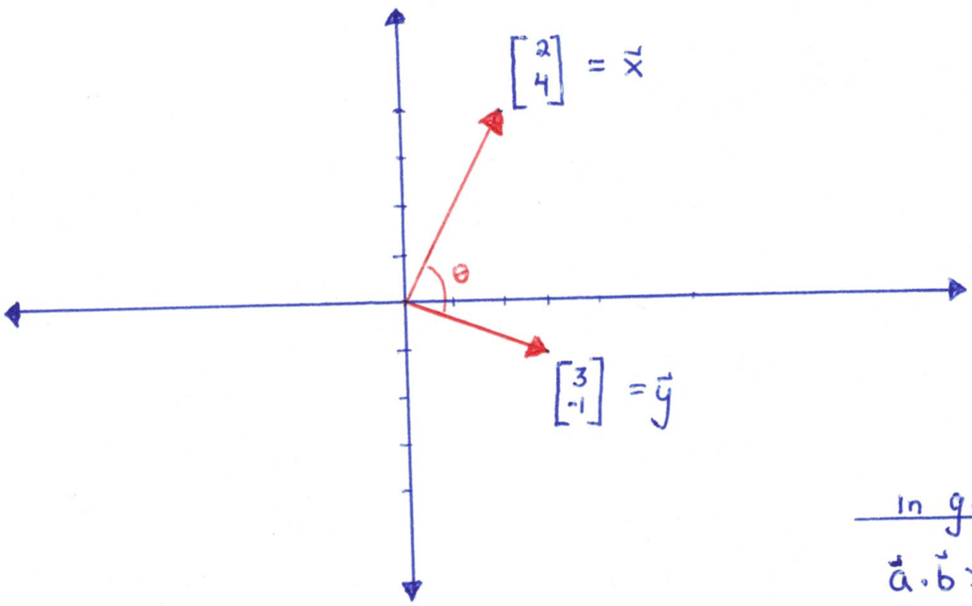
$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times 1}, \qquad \vec{y}_1 = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_{2\times 1}$$

$$\vec{x}\cdot\vec{y} = x_1 y_1 + x_2 y_2 = \sum_{i=1}^{2} x_i y_i$$

Dot Product in $\mathbb{R}^3$: If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{3\times 1}$, $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}_{3\times 1}$,

then $\vec{x}\cdot\vec{y} = x_1 y_1 + x_2 y_2 + x_3 y_3 = \sum_{i=1}^{3} x_i y_i$

Example 11.3.1 p. 782

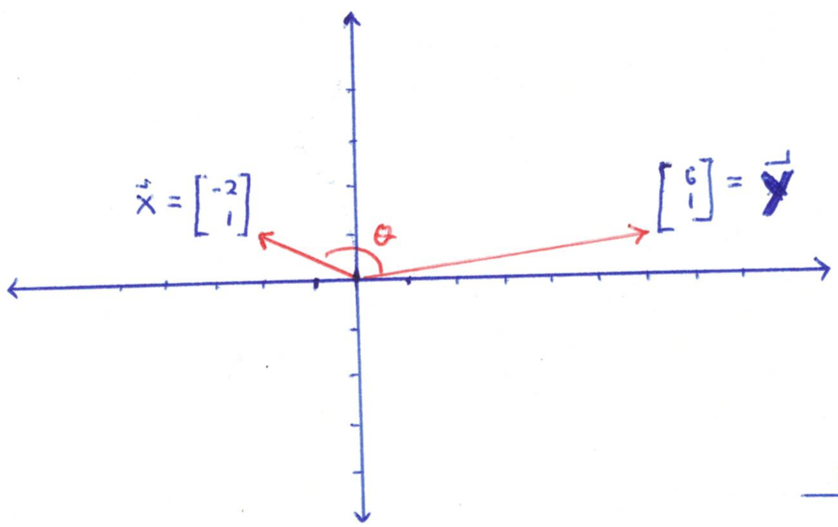


$$\vec{x} \cdot \vec{y} = 2 \cdot 3 + 4 \cdot -1$$
$$= 6 - 4$$
$$= 2.$$

In general

$\vec{a} \cdot \vec{b} > 0$ iff θ accute

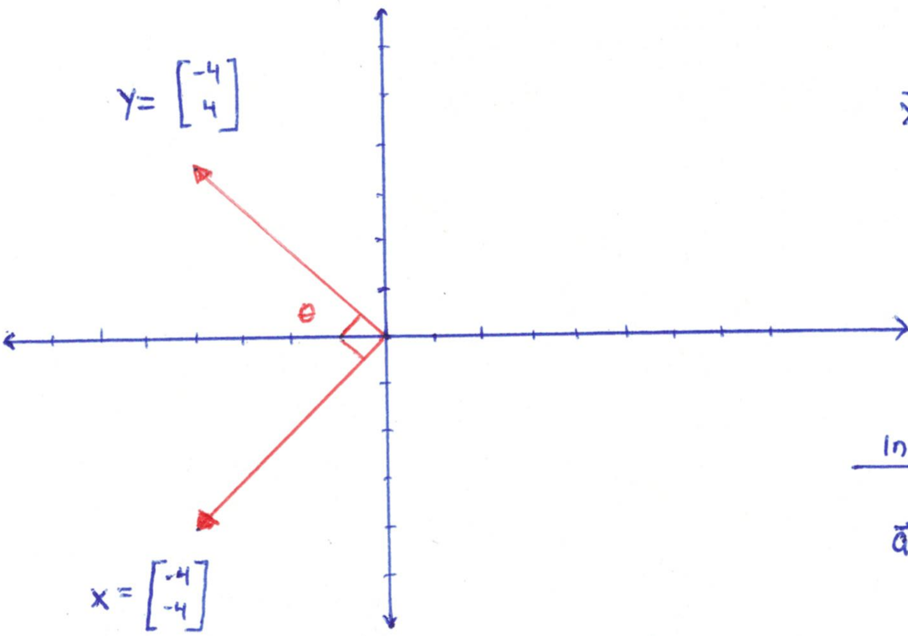
□ Why? Where does this property come from?



$$\vec{x} \cdot \vec{y} = -2 \cdot 6 + 1 \cdot 1$$
$$= -12 + 1$$
$$= -11$$

In general:

$\vec{a} \cdot \vec{b} < 0$ iff θ obtuse



$$\vec{x} \cdot \vec{y} = -4 \cdot -4 + 4 \cdot -4$$
$$= +16 + -16$$
$$= 0$$

<u>In general:</u>

$\vec{a} \cdot \vec{b} = 0$ iff $\theta = \pi/2$

These are the <u>algebraic</u> properties of dot product!

<u>Properties of Dot Product</u> p. 651

1. Length via dot product: $\underbrace{\|\vec{x}\|_2^2}_{\text{"the 2-norm" of } \vec{x} \text{ squared}} = \underbrace{\vec{x} \cdot \vec{x}}_{\text{"}\vec{x} \text{ dot } \vec{x}\text{"}}$

- If $\vec{x} \in \mathbb{R}^2$, $\|\vec{x}\|_2 = \sqrt{x_1^2 + x_2^2}$
- If $\vec{x} \in \mathbb{R}^3$, $\|\vec{x}\|_2 = \sqrt{x_1^2 + x_2^2 + x_3^2}$
- If $\vec{x} \in \mathbb{R}^4$, $\|\vec{x}\|_2 = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}$

Let $\vec{x} \in \mathbb{R}^3$

$$\Rightarrow \|x\|_2^2 = \left(\sqrt{x_1^2 + x_2^2 + x_3^2}\right)^2 = x_1^2 + x_2^2 + x_3^2$$

$$= x_1 \cdot x_1 + x_2 \cdot x_2 + x_3 \cdot x_3 = \vec{x} \cdot \vec{x} \quad \square$$

2. Commutativity of dot product: $\vec{x}\cdot\vec{y} = \vec{y}\cdot\vec{x}$ ← (also Known as symmetry)

Let $\vec{x}, \vec{y} \in \mathbb{R}^3$. Then $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$.

Consider $\vec{x}\cdot\vec{y} = x_1 y_1 + x_2 y_2 + x_3 y_3$

$= y_1 x_1 + y_2 x_2 + y_3 x_3$

$= \vec{y}\cdot\vec{x}$. $\square$

3. Linearity of dot product: $\vec{x}\cdot(\vec{y}+\vec{z}) = \vec{x}\cdot\vec{y} + \vec{x}\cdot\vec{z}$

Let $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^3$ with $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, $\vec{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$.

Consider $\vec{x}\cdot(\vec{y}+\vec{z}) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} y_1+z_1 \\ y_2+z_2 \\ y_3+z_3 \end{bmatrix}$

$= x_1(y_1+z_1) + x_2(y_2+z_2) + x_3(y_3+z_3)$

$= x_1 y_1 + x_1 z_1 + x_2 y_2 + x_2 z_2 + x_3 y_3 + x_3 z_3$

$= (x_1 y_1 + x_2 y_2 + x_3 y_3) + (x_1 z_1 + x_2 z_2 + x_3 z_3)$

$= \vec{x}\cdot\vec{y} + \vec{x}\cdot\vec{z}$. $\square$

Properties of the dot Product (continued) p. 651

4. Homogeneity of dot product: $\alpha(\vec{x}\cdot\vec{y}) = (\alpha\vec{x})\cdot\vec{y} = \vec{x}\cdot(\alpha\vec{y})$

Let $\alpha \in \mathbb{R}$, $\vec{x},\vec{y} \in \mathbb{R}^3$ with $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$.

Consider $\alpha(\vec{x}\cdot\vec{y}) = \alpha(x_1y_1 + x_2y_2 + x_3y_3)$

$= (\alpha x_1)y_1 + (\alpha x_2)y_2 + (\alpha x_3)y_3$

$= (\alpha\vec{x})\cdot\vec{y}$

$= x_1(\alpha y_1) + x_2(\alpha y_2) + x_3(\alpha y_3)$

$= \vec{x}\cdot(\alpha\vec{y})$. ▫

5. Dot products w/ zero: $\vec{x}\cdot\vec{0} = 0$

Let $\vec{x}, \vec{0} \in \mathbb{R}^3$ with $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$\Rightarrow \vec{x}\cdot\vec{0} = x_1\cdot 0 + x_2\cdot 0 + x_3\cdot 0 = 0$ ✓

This is a geometric property of the dot product

The cosine formula for the dot product p. 649

The dot product of two non zero vectors $\vec{x}, \vec{y}$ is given by

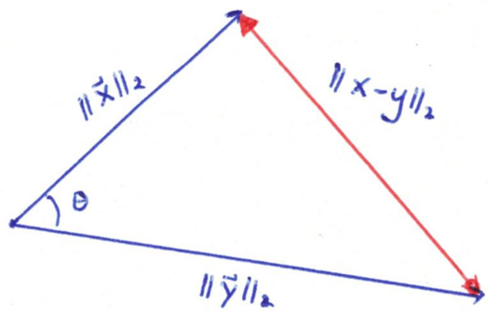
$$\boxed{\vec{x} \cdot \vec{y} = \|\vec{x}\|_2 \cdot \|y\|_2 \cos(\theta)}$$

where θ is the angle between $\vec{x}$ and $\vec{y}$, $0 \leq \theta \leq \pi$.

case 1: $0 < \theta < \pi$ (ie $\vec{x} \neq \alpha \vec{y}$ for all $\alpha \in \mathbb{R}$)

Proof: Let $\vec{x}, \vec{y}$ be non-zero vectors

Assume $0 < \theta < \pi$



By the law of cosines, we know

$$\|\vec{x} - \vec{y}\|_2^2 = \|\vec{x}\|_2^2 + \|\vec{y}\|_2^2 - 2\|\vec{x}\|_2 \|\vec{y}\|_2 \cos(\theta) \quad \boxed{\mathrm{I}}$$

looking back at the algebraic properties of the dot product, we have

$$\|\vec{x}-\vec{y}\|_2^2 = (\vec{x}-\vec{y})\cdot(\vec{x}-\vec{y})$$

$$= (\vec{x}-\vec{y})\cdot\vec{x} - (\vec{x}-\vec{y})\cdot\vec{y}$$

$$= \vec{x}\cdot\vec{x} - \vec{y}\cdot\vec{x} - \vec{x}\cdot\vec{y} + \vec{y}\cdot\vec{y}$$

$$= \|\vec{x}\|_2^2 + \|\vec{y}\|_2^2 - 2\,\vec{x}\cdot\vec{y} \qquad \boxed{II}$$

Then by $\boxed{I}$ and $\boxed{II}$ we have

$$\|x\|_2^2 + \|\vec{y}\|_2^2 - 2\,\vec{x}\cdot\vec{y} = \|\vec{x}\|_2^2 + \|\vec{y}\|_2^2 - 2\,\|x\|_2\|y\|_2\cos(\theta)$$

$$\Rightarrow \quad -2\,\vec{x}\cdot\vec{y} = -2\,\|\vec{x}\|_2\|\vec{y}\|_2\cos(\theta)$$

$$\Rightarrow \quad \boxed{\vec{x}\cdot\vec{y} = \|x\|_2\,\|\vec{y}\|_2\cos(\theta)}\,. \blacksquare$$

case 2: Assume $\theta = 0$ or $\theta = \pi$. In other words, suppose

$$\vec{y} = \alpha \vec{x} \quad \text{for some } \alpha \in \mathbb{R}$$

if $\alpha > 0 \Rightarrow \cos(\theta) = 1 = \mathrm{sgn}(\alpha), \ (\theta = 0)$

if $\alpha < 0 \Rightarrow \cos(\theta) = -1 = \mathrm{sgn}(\alpha), (\theta = \pi)$

Then consider

$$\vec{x} \cdot \vec{y} = \vec{x} \cdot \alpha \vec{x}$$

$$= \alpha \ \vec{x} \cdot \vec{x}$$

$$= \alpha \ \|\vec{x}\|_2^2$$

$$= (\alpha \cdot \|\vec{x}\|_2) \cdot (\|\vec{x}\|_2)$$

$$= \underbrace{\mathrm{sgn}(\alpha) \cdot |\alpha| \ \|x\|_2} \cdot \|x\|_2$$

$$= \mathrm{sgn}(\alpha) \cdot \|\alpha \vec{x}\|_2 \cdot \|\vec{x}\|_2$$

$$= \mathrm{sgn}(\alpha) \cdot \|\vec{y}\|_2 \cdot \|x\|_2$$

$$= \boxed{\|x\|_2 \ \|y\|_2 \ \cos(\theta)} \quad \blacksquare$$

Def: Two nonzero vectors $\vec{x}, \vec{y}$ are orthogonal if the angle between these vectors is $\theta = \pi/2$

Thm: Two vectors $\vec{x}$ and $\vec{y}$ are orthogonal if and only if

$$\vec{x} \cdot \vec{y} = 0$$

Proof. $\vec{x} \cdot \vec{y} = \|x\|_2 \cdot \|\vec{y}\|_2 \underbrace{\cos(\theta)}_{0 \text{ when } \theta = \pi/2}$

Example 9.1.4: Show that $\vec{x} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ is perpendicular to $\begin{bmatrix} 5 \\ -4 \\ 2 \end{bmatrix} = \vec{y}$.

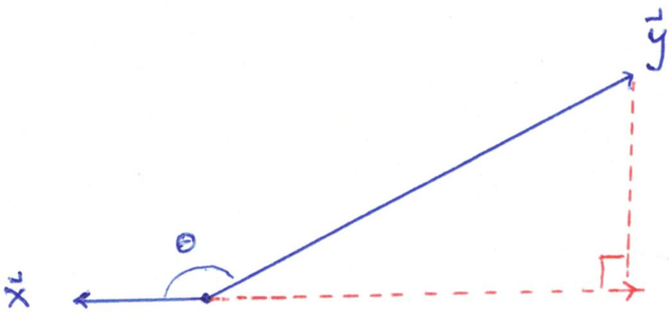
Consider $\vec{x} \cdot \vec{y} = 2 \cdot 5 + 2 \cdot -4 + -1 \cdot 2$

$= 10 - 8 - 2$

$= 10 - 10$

$= 0$. ✓

Projections in general



$\vec{z} = \vec{y} - \vec{p}$

$\vec{p} = \text{proj}_{\vec{x}}(\vec{y}) = \alpha \vec{x}$

- How do we find this scalar α to construct $\vec{p}$?

consider

$$\vec{x} \cdot \vec{z} = \vec{x} \cdot (\vec{y} - \vec{p})$$

$$= \vec{x} \cdot (\vec{y} - \alpha \vec{x})$$

$$= \vec{x} \cdot \vec{y} - \alpha \, \vec{x} \cdot \vec{x}$$

We want to force $\vec{z}$ orthogonal to $\vec{x}$, which occurs iff

$$\vec{x} \cdot \vec{z} = 0 = \vec{x} \cdot \vec{y} - \alpha \, \vec{x} \cdot \vec{x}$$

$$\Rightarrow \quad \alpha \, \vec{x} \cdot \vec{x} = \vec{x} \cdot \vec{y}$$

$$\Rightarrow \quad \alpha = \frac{\vec{x} \cdot \vec{y}}{\vec{x} \cdot \vec{x}} = \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\|_2^2} = \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\|_2} \cdot \frac{1}{\|\vec{x}\|_2}$$

Then, we define

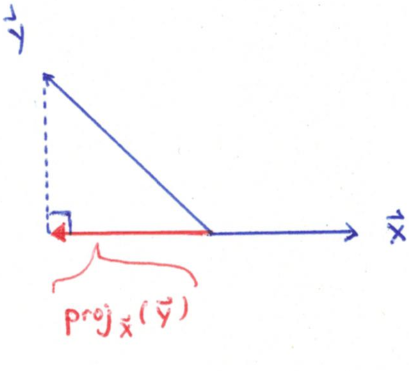
Vector projection of $\vec{y}$ onto $\vec{x}$ $\longrightarrow$ $\vec{p} = \text{proj}_{\vec{x}}(\vec{y})$

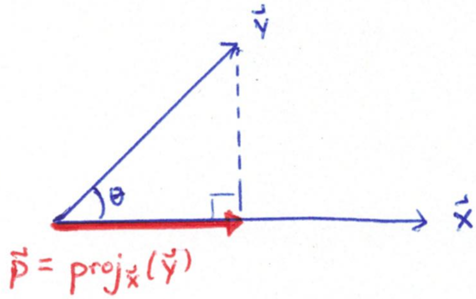
$$= \alpha \, \vec{x}$$

$$= \underbrace{\frac{\vec{x} \cdot \vec{y}}{\|x\|_2}}_{\substack{\text{scalar projecton of } \vec{y} \text{ onto } \vec{x} \\ \text{(component of } \vec{y} \\ \text{along } \vec{x}\text{)}}} \cdot \underbrace{\frac{\vec{x}}{\|\vec{x}\|}}_{\substack{\text{Unit vector} \\ \text{(length 1)} \\ \text{in the direction of } \vec{x}}}$$

Scalar component of $\vec{y}$ in the direction of $\vec{x}$

$$\text{Scal}_{\vec{x}}(\vec{y}) = \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\|_2}$$



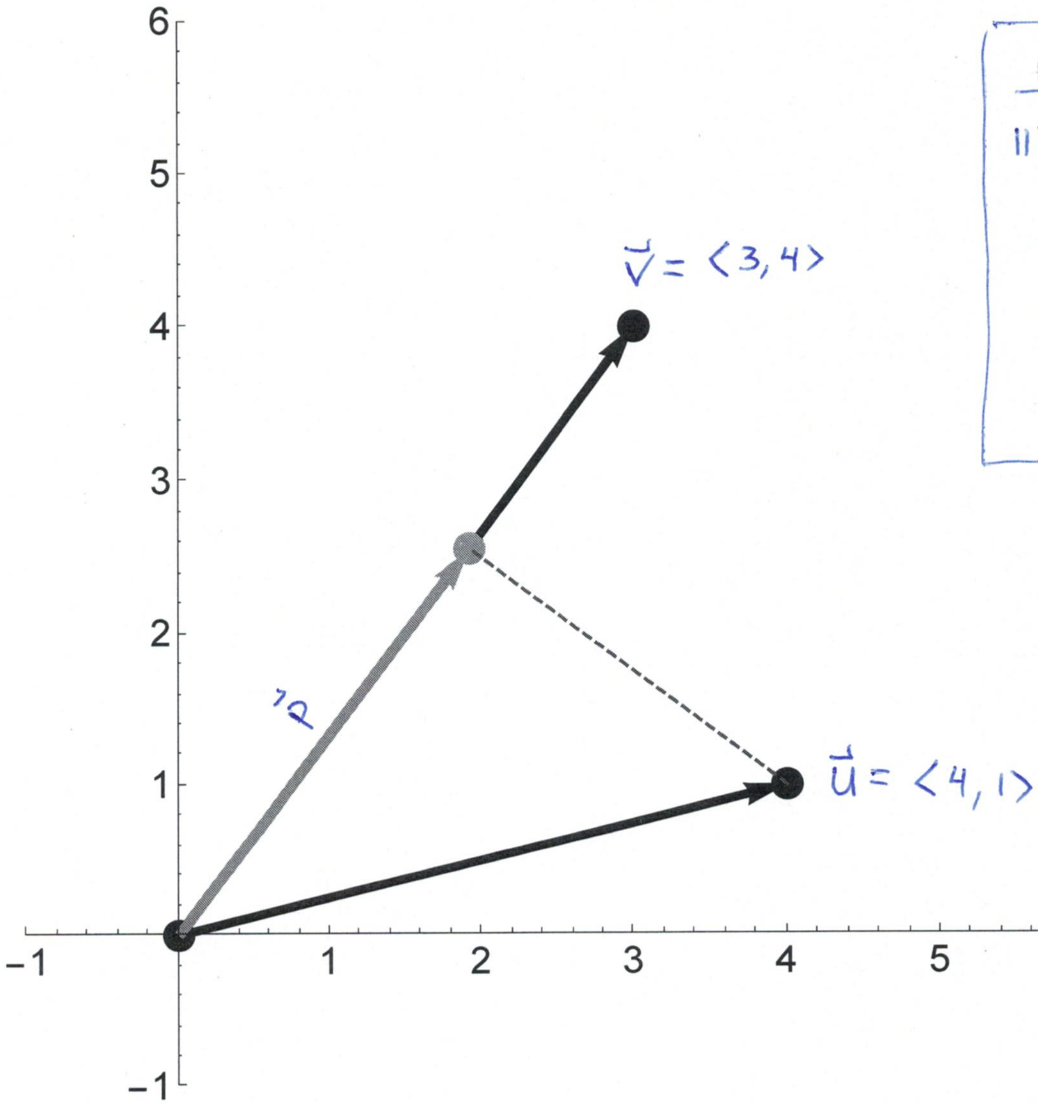


$$\|\vec{p}\| = \|\vec{y}\|_2 \cos(\theta)$$

$$= \|\vec{y}\|_2 \cdot \frac{\vec{x} \cdot \vec{y}}{\|x\|_2 \|y\|_2}$$

$$= \frac{\vec{x} \cdot \vec{y}}{\|x\|_2} \boxed{\neq \text{comp}_{\vec{x}}(\vec{y})}$$

Example 11.3.3a p. 786



Note:

$$\|\vec{v}\|_2 = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5 \checkmark$$

$$\vec{p} = \text{Proj}_{\vec{v}}(\vec{u})$$

$$= \underbrace{\left[\frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|_2}\right]}_{\text{Scal}_{\vec{v}}(\vec{u})} \cdot \underbrace{\frac{\vec{v}}{\|\vec{v}\|_2}}_{\text{Unit vector in direction of } \vec{v}}$$

$$= \left[\frac{\langle 4, 1 \rangle \cdot \langle 3, 4 \rangle}{5}\right] \cdot \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$$

$$= \left[\frac{4 \cdot 3 + 1 \cdot 4}{5}\right] \cdot \left\langle \frac{3}{5}, \frac{4}{5}\right.$$

$$= \frac{16}{5} \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$$

$$= \boxed{\frac{16}{25} \langle 3, 4 \rangle}$$