Lesson 4: Cross Products

- ☐ Study Skills HW 4 due today
- ☐ Last Day to Drop for full refund:

  Friday of week 2

- ☐ Warm up activity
- ☐ Lesson 4 Lecture content

Operations on Vectors in $\mathbb{R}^3$

Vector - vector addition: $+ : \underbrace{\mathbb{R}^3 \times \mathbb{R}^3}_{\text{domain space}} \longrightarrow \underbrace{\mathbb{R}^3}_{\text{codomain}}$

domain space (input two vectors, each of size 3x1)

codomain (output one vector of size 3x1)

- useful to combine vectors with different direction

scalar - vector multiplication: $\cdot : \underbrace{\mathbb{R} \times \mathbb{R}^3}_{\text{domain space}} \longrightarrow \underbrace{\mathbb{R}^3}_{\text{codomain}}$

domain space (input one scalar and one 3x1 vector)

codomain (output one 3x1 vector)

- useful to stretch, compress, invert a vector while maintaining same direction

dot product: $\cdot : \underbrace{\mathbb{R}^3 \times \mathbb{R}^3}_{\text{domain space}} \longrightarrow \underbrace{\mathbb{R}}_{\text{codomain}}$

domain space (input two vectors, each of size 3x1)

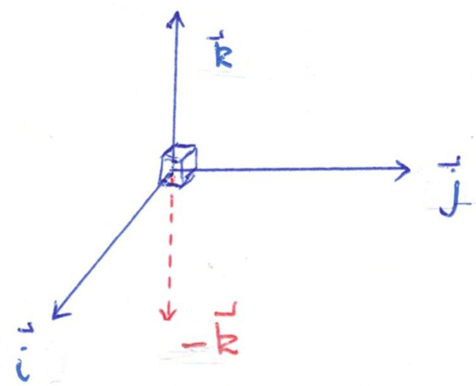
codomain (output one scalar)

- measurement of "parallelity" between vectors
- useful to project one vector onto another

Today we study another operation: the cross product

<u>Example 2 p. 656</u> Find the normal vectors (orthogonal vectors) to each of the standard basis vector pairs.
(Be sure to highlight "right-hand" rule"

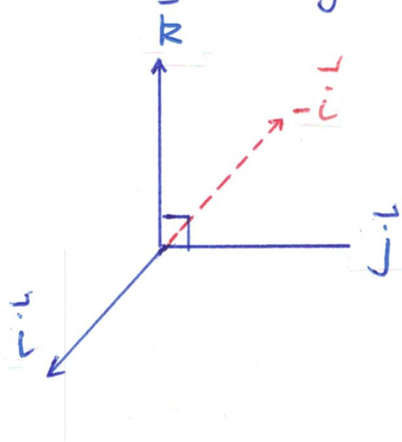
□ What vectors in $\mathbb{R}^3$ are orthogonal to both $\vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\vec{j} =$



$$\vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \vec{i} \times \vec{j}$$

OR $$-\vec{k} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \vec{j} \times \vec{i}$$

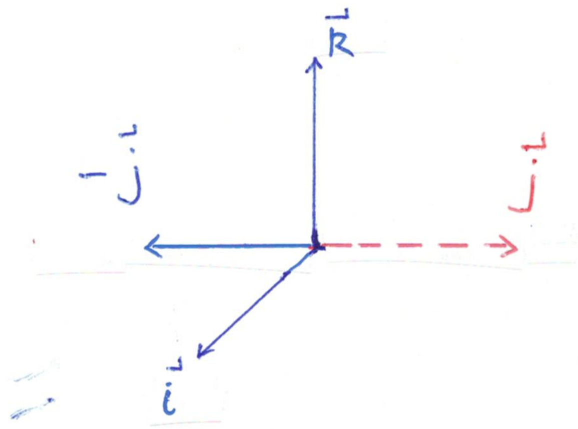
□ What vectors in $\mathbb{R}^3$ are orthogonal to both $\vec{j}$ and $\vec{k}$



$$\vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \vec{j} \times \vec{k}$$

OR $$-\vec{i} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \vec{k} \times$$

□ What vectors in $\mathbb{R}^3$ are orthogonal to both $\vec{i}$ and $\vec{k}$?



$$-\vec{j} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \vec{i} \times \vec{k}$$

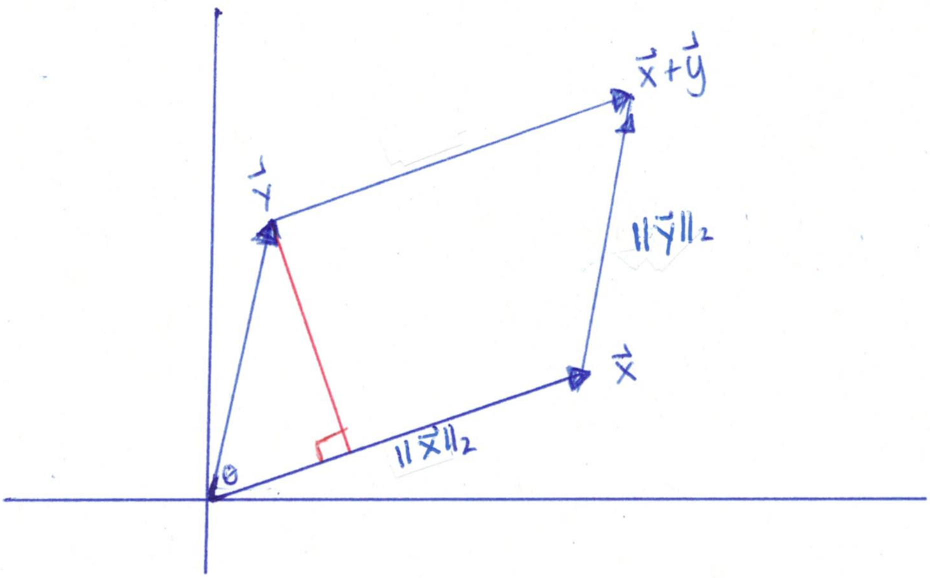
or $$\vec{j} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \vec{k} \times \vec{i}$$

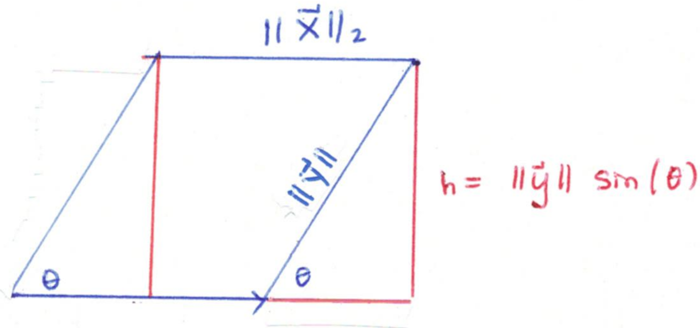
(put a negative on 2nd component of normal to maintain orientatio

Note: Make sure to use Right hand rule while evaluating these cross products.

suppose we wish to construct a measurement of perpendicularity between two vectors $\vec{x}$ and $\vec{y}$.

In particular, consider the diagram:



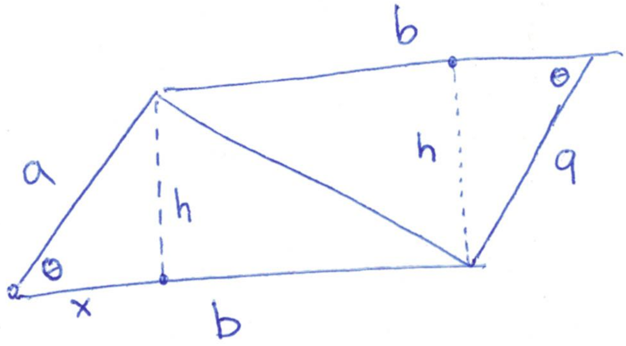


Area of parallelogram $= \boxed{\|\vec{x}\| \cdot \underbrace{\|\vec{y}\| \sin(\theta)}}$

length of vector x

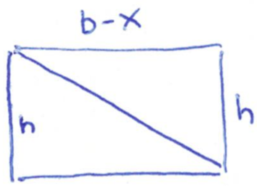
the "amount of $\vec{y}$" perpendicular to vector $\vec{x}$

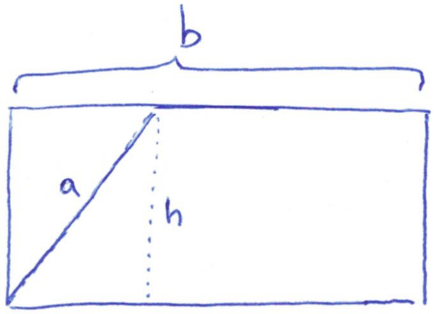
Proof of Sine formula for Cross product



Note: $h = a \cdot \sin(\theta)$ and $x = a\cos(\theta)$

Area of Rectangle:





Total area $= h \cdot b$

$= a\sin(\theta) \cdot b$

$= \boxed{a \cdot b \cdot \sin(\theta)}$!!

Notice: We will have a nonzero area iff there is a piece of $\vec{y}$ that is perpendicular to $\vec{x}$ (i.e. $\|\vec{y}\|_2 \sin(\theta) \neq 0 \iff \theta \neq 0$)

Major intuition behind cross product

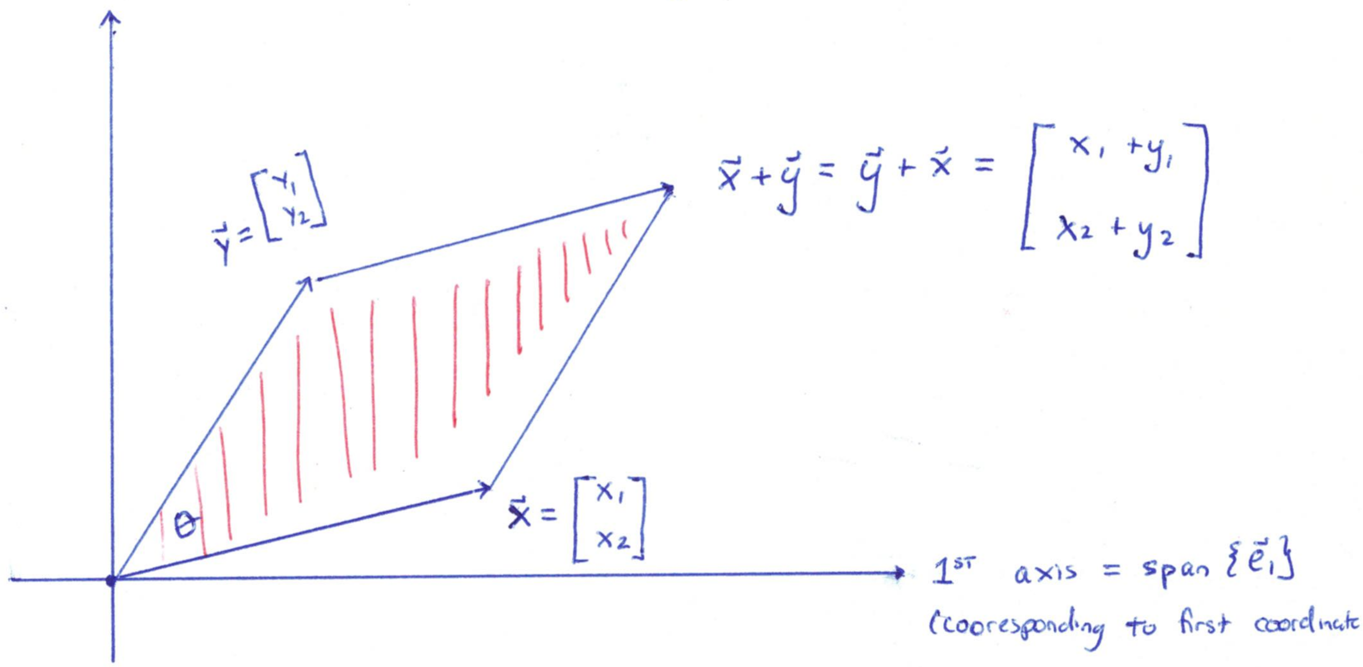
Suppose we are given two vectors $\vec{x}, \vec{y} \in \mathbb{R}^2$.

Suppose we want to measure

how much is $\vec{y}$ perpendicular to $\vec{x}$

coorresponding to 2nd coordinate

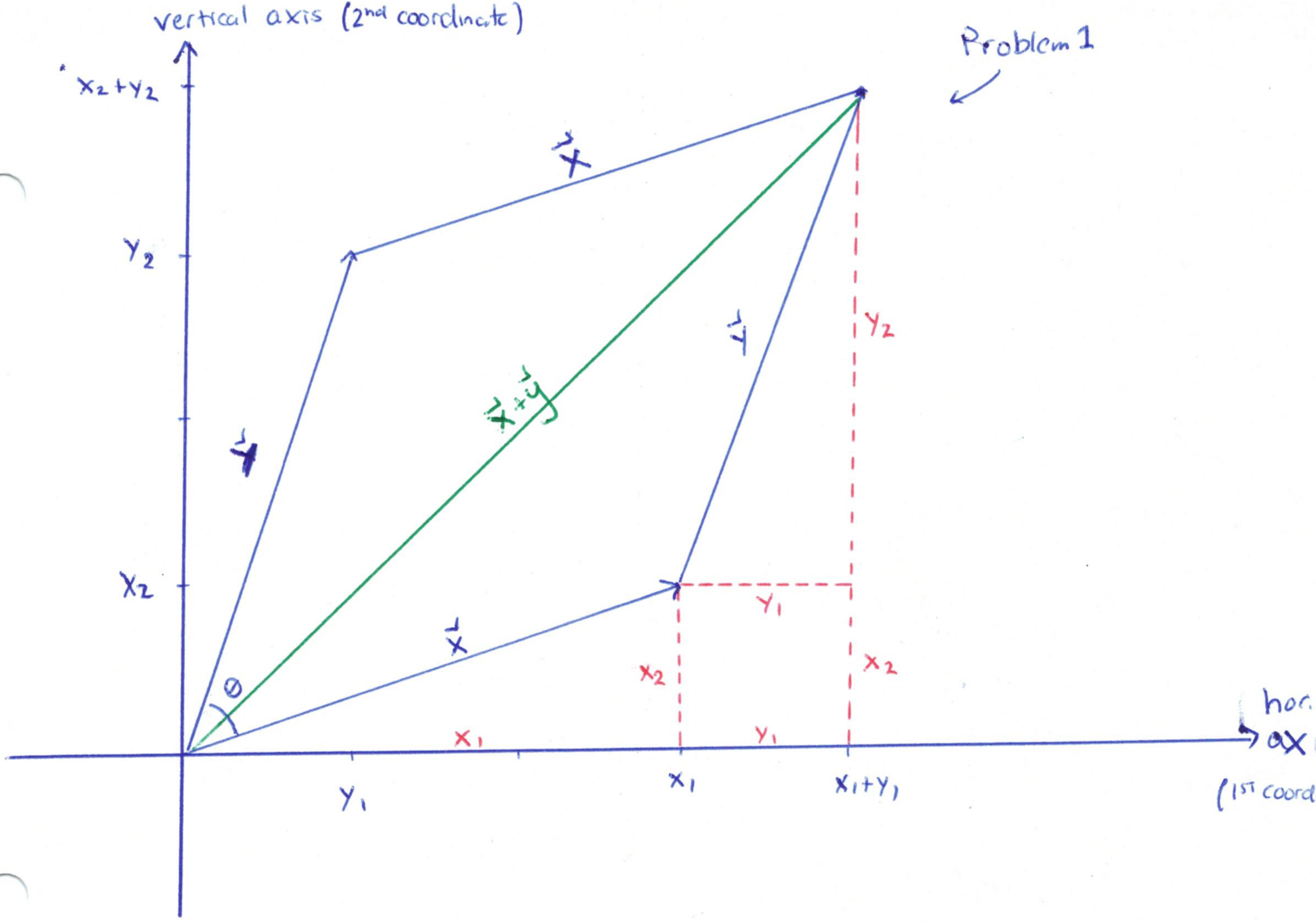
$\text{span}\{\vec{e}_2\} = 2^{nd}$ axis



Key observation

- The more "perpendicular" $\vec{y}$ is to $\vec{x}$, the greater the area of the parallelogram

- The less "perpendicular" $\vec{y}$ is to $\vec{x}$ the smaller the area of the parallelogram

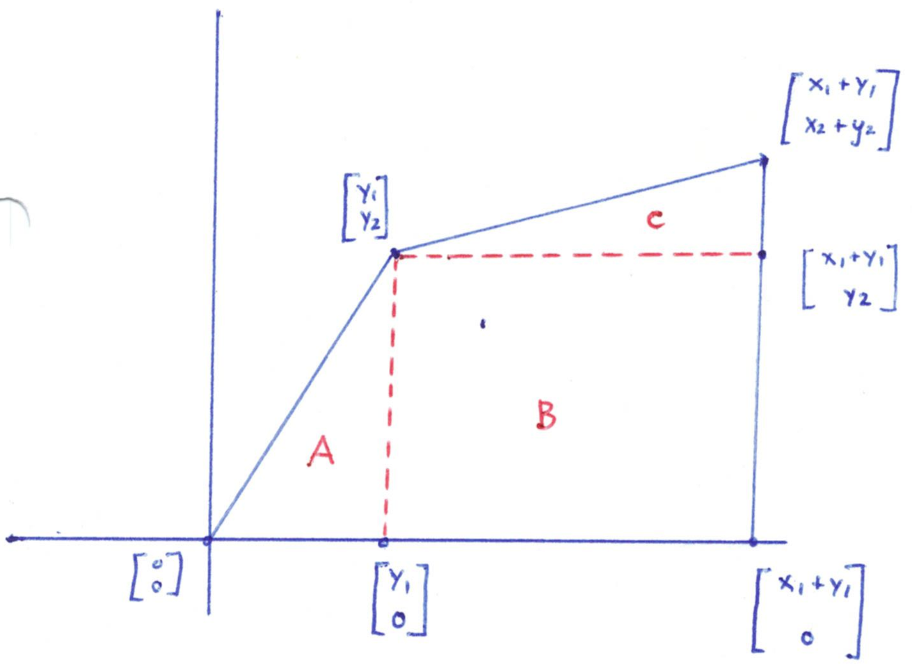


Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

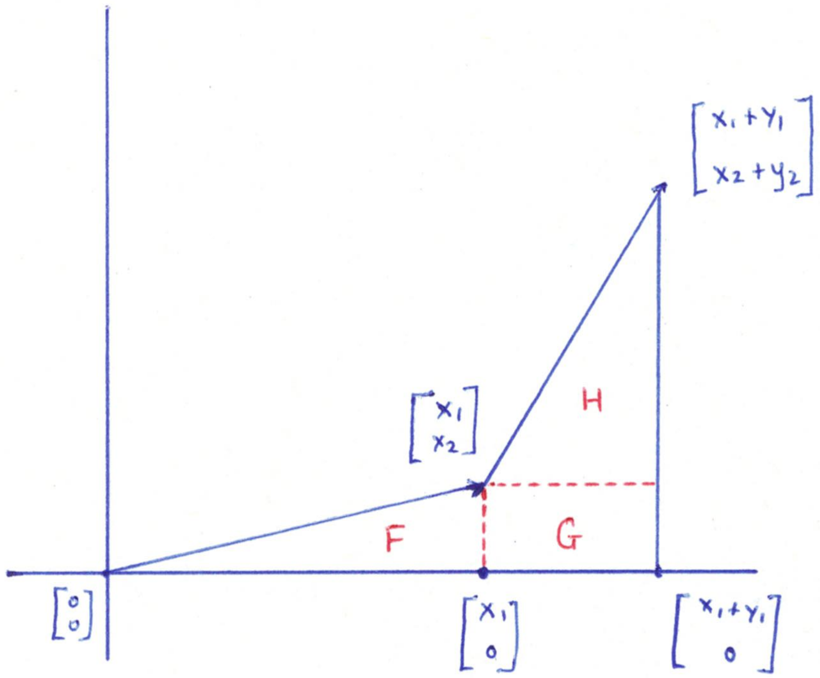
Then $\vec{x}+\vec{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix} = \vec{y}+\vec{x}$.

Thus, to measure how much $\vec{y}$ is perpendicular to $\vec{x}$, we can use the concept of the area of the parallelogram formed by $\vec{x}$ and $\vec{y}$.

Let's take a look at how we might calculate this area



$$\begin{bmatrix} \text{Total Area} \\ \text{of shape} \\ A+B+C \end{bmatrix} = \underbrace{\frac{1}{2}\, y_1 y_2}_{\substack{\text{Area of} \\ \text{triangle A} \\ \text{is } \frac{1}{2}\cdot b_A \cdot h_A}} + \underbrace{x_1 \cdot y_2}_{\substack{\text{Area of} \\ \text{square B} \\ \text{is } b_B \cdot h_B}} + \underbrace{\frac{1}{2}\, x_1 \cdot x_2}_{\substack{\text{Area of} \\ \text{triangle C} \\ \text{is } \frac{1}{2}\cdot b_C \cdot h_C}}$$



Total area of shape

$$F + G + H = \underbrace{\frac{1}{2} x_1 \cdot x_2}_{\text{Area of triangle F is } \frac{1}{2} b_F h_F} + \underbrace{y_1 x_2}_{\text{Area of square G is } b_G h_G} + \underbrace{\frac{1}{2} y_1 y_2}_{\text{Area of triangle H is } \frac{1}{2} b_H h_H}$$

Notice that the total area of parallelogram will be

$$(Area_A + Area_B + Area_C) - (Area_F + Area_G + Area_H)$$

$$= \frac{1}{2} y_1 y_2 + x_1 y_2 + \frac{1}{2} x_1 x_2 - \frac{1}{2} x_1 x_2 - x_2 y_1 - \frac{1}{2} y_1 y_2$$

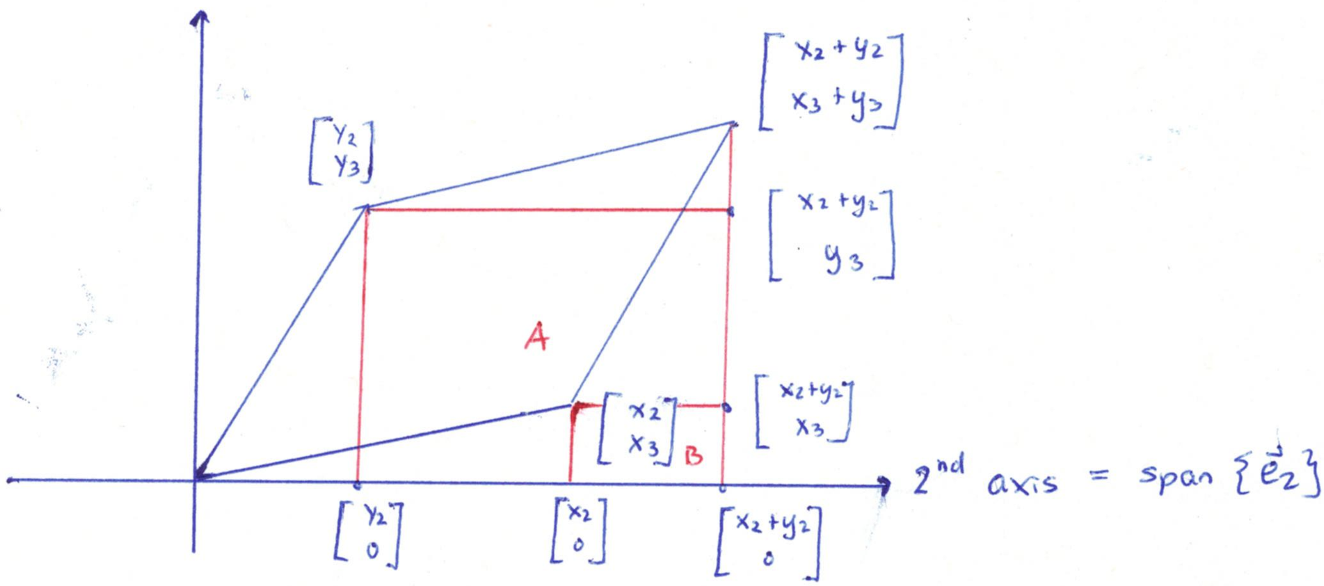
Area of parallelogram : $\boxed{x_1 y_2 - x_2 y_1} = \text{area}_{12}$

in dimensions 1 & 2

Measurement of how much $\vec{y}$ is perpendicular to $\vec{x}$

Now, what if we used other coordinate axises

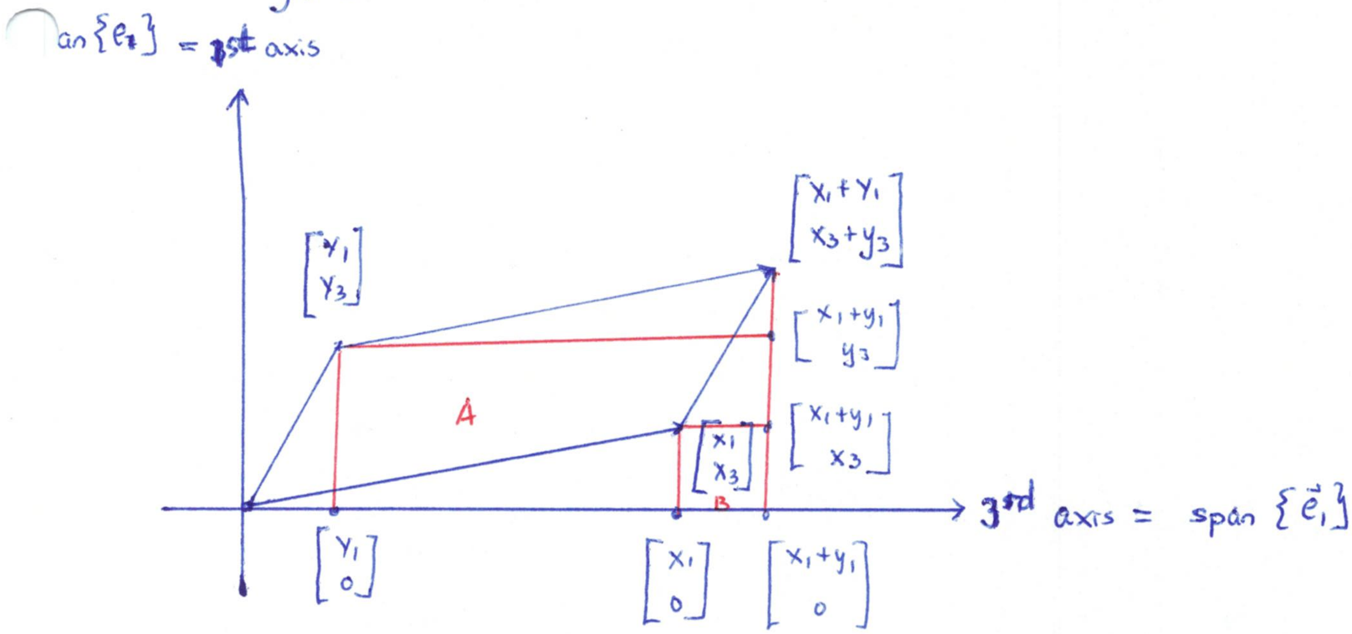
span$\{\vec{e}_3\}$ = 3rd axis



Area of parallelogram = Area square A − Area square B

$$\boxed{x_2 y_3 - y_2 x_3 = \text{area}_{23}}$$

Finally, what if we use the last two axis



Area of parallelogram = Area square A − Area square B

$$\boxed{x_1 y_3 - y_1 x_3 = \text{area}_{13}}$$

Now, this gives us a method for measuring how perpendicular

are the vectors

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

by measuring the "degree" of perpendicularity (area of parallelogram) for each two coordinate axis

L4 p12

Although we have our major source of intuition (measure perpendicularity using area of parallelogram) and although the basic premise of our calculations are the same in each case, we need a way to track which area goes with which parallelogram :

$\text{area}_{12} = x_1 y_2 - x_2 y_1 \longleftarrow +\vec{k}$ vector is perpendicular to $\vec{i}$ and $\vec{j}$

$\text{area}_{23} = x_2 y_3 - y_2 x_3 \longleftarrow +\vec{i}$ vector is perpendicular to $\vec{j}$ and $\vec{k}$

$\text{area}_{13} = x_1 y_3 - x_3 y_1 \longleftarrow -\vec{j}$ vector is perpendicular to $\vec{i}$ and $\vec{k}$

To encode this distinction, we'll use a vector perpendicular to the chosen parallelogram

Then we define

$$\tilde{x} \times \tilde{y} = \begin{bmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{bmatrix}$$

$$= (x_2 y_3 - x_3 y_2) \cdot \vec{i}$$

$$+ (x_3 y_1 - x_1 y_3) \cdot \vec{j}$$ ← negative factored in.

$$+ (x_1 y_2 - x_2 y_1) \cdot \vec{k}$$

Cross Product Definition

Let $\vec{x}, \vec{y} \in \mathbb{R}^3$. Then the cross product between $\vec{x}$ and $\vec{y}$ is ORTHOGONAL to both $\vec{x}$ and $\vec{y}$.

It can be found using one of two formulas

$$\vec{x} \times \vec{y} = \begin{bmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{bmatrix}$$

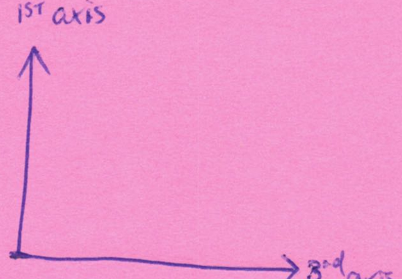
← cross product in component form

$$= \|\vec{x}\| \cdot \|\vec{y}\| \sin(\theta) \cdot \vec{n}$$

← Cross product in geometric form

where $\vec{n}$ = unit vector orthogonal to both $\vec{x}$ & $\vec{y}$.

Be careful here to switch orientation



Results for Cross Product

- Two non zero vectors are parallel $\Leftrightarrow \vec{x} \times \vec{y} = \vec{0}$

- $\vec{x} \cdot (\vec{x} \times \vec{y}) = 0 = \vec{y} \cdot (\vec{x} \times \vec{y})$ since $\vec{x} \times \vec{y} \perp \vec{x}$ and $\vec{x} \times \vec{y} \perp \vec{y}$

- Algebraic properties of Cross product

  all can be proved using component form.

  Let $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^3$, $\alpha \in \mathbb{R}$. Then

  1. $\vec{x} \times \vec{y} = -\vec{y} \times \vec{x}$
  2. $(\alpha \vec{x}) \times \vec{y} = \alpha (\vec{x} \times \vec{y}) = \vec{x} \times (\alpha \vec{y})$
  3. $\vec{x} \times (\vec{y} + \vec{z}) = \vec{x} \times \vec{y} + \vec{x} \times \vec{z}$
  4. $(\vec{x} + \vec{y}) \times \vec{z} = \vec{x} \times \vec{z} + \vec{y} \times \vec{z}$

- Length $\|\vec{x} \times \vec{y}\|_2 = \big\| \, \|\vec{x}\|_2 \cdot \|y\|_2 \sin(\theta) \, \vec{n} \, \big\|_2$

$$= \|x\|_2 \cdot \|y\|_2 \, |\sin(\theta)| \cdot \|\vec{n}\|_2$$

has unit length

$$= \|\vec{x}\|_2 \, \|\vec{y}\|_2 \, |\sin(\theta)|$$

area of parallelogram formed by $\vec{x}$ and $\vec{y}$. $\boxed{\text{L4 p}}$

Using Determinants to calculate the Cross Product

□ show cofactor method and Rule of Sarrus method.

Example: Find the cross product of $\vec{x} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 2 \\ 7 \\ -5 \end{bmatrix}$

$$\vec{x} \times \vec{y} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} = \begin{vmatrix} i & j & k \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix}$$

these lines represent the determinant of a 3x3 matrix