# Lesson 5: Lines and Planes

## Lines in $\mathbb{R}^2$:

- Start with two points

  $P(x_1, y_1)$ AND $Q(x_2, y_2)$

- Recall the equations for a line

  Point-Slope form: $y - y_1 = m(x - x_1)$

  Slope-Intercept form: $y = mx + b$

  Standard form: $Ax + By + C = 0$

Example 1: Consider the line defined by the two points

$(0, -2)$ and $(2, -3)$

$x_1 = 0,\ y_1 = -2 \qquad x_2 = 2,\ y_2 = -3$

We see $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-3 - -2}{2 - 0} = \dfrac{-3+2}{2} = \dfrac{-1}{2}$

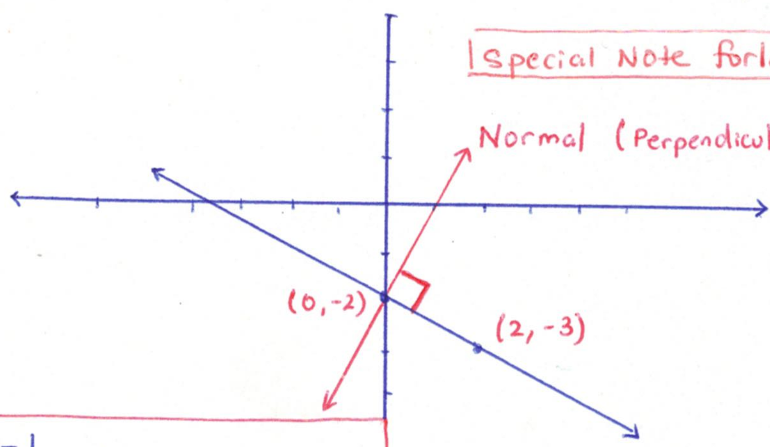
$\Rightarrow (y - -2) = m(x - 0)$ ← point-slope form of a line

$\Rightarrow y + 2 = -\frac{1}{2}x$

$\Rightarrow y = -\frac{1}{2}x - 2$ ← slope-intercept form

$\Rightarrow 2y = -1x - 4$

$\Rightarrow 1x + 2y + 4 = 0$ ← $\boxed{\text{Standard form of a line}}$




$\boxed{\text{Special Note for later}}$

Normal (Perpendicular) line w/ slope

$-\dfrac{1}{-1/2} = \dfrac{2}{1} = \dfrac{B}{A}$

$\boxed{\text{slope: } m = \dfrac{-A}{B} = \dfrac{-1}{2} \Rightarrow A = 1,\ B = 2}$

Example 1 ...

Notice, the slope of this line is given by

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1}{2} = -\frac{A}{B}$$

Let's introduce notation to explicitly connect the point-slope form with standard form

$$\Rightarrow \boxed{y - y_1 = m(x - x_1)}$$ point-slope form

$$\Rightarrow y - y_1 = \frac{-A}{B}(x - x_1)$$

$$\Rightarrow B(y - y_1) = -A(x - x_1)$$

$$\Rightarrow By - By_1 = -Ax + Ax_1$$

$$\Rightarrow Ax + By - Ax_1 - By_1 = 0$$

$$\Rightarrow \boxed{Ax + By + C = 0}$$ where $C = -Ax_1 - By_1$

standard form

Example 1 ...

What if we want to create a vector equation for our line through points $(0,-2)$ and $(2,-3)$?

In other words, what if we consider

$$\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} t \\ -\tfrac{1}{2}t - 2 \end{bmatrix}$$

$t$ is called a parameter

$$= \begin{bmatrix} 1 \\ -1/2 \end{bmatrix} t + \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$= \vec{v} \cdot t + \vec{r}_0$$

direction of line

point on line

Notice: $\vec{r}(t)$ is a vector-valued function

$$\vec{r} : \mathbb{R} \longrightarrow \mathbb{R}^2$$

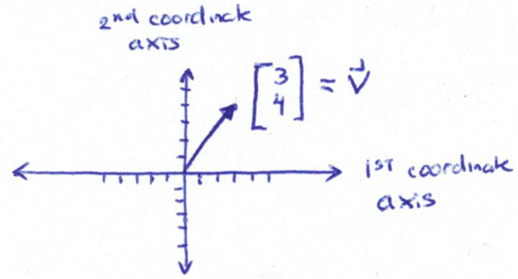
2D output (vector valued)

## Example

Using Vector Addition & Scalar Multiplication to Create lines in $\mathbb{R}^2$

Suppose I give you a vector

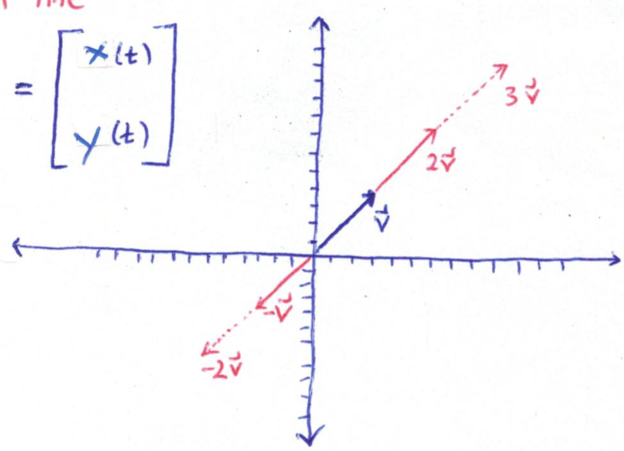
$$\vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$




□ How might you use this vector to create a line in $\mathbb{R}^2$

parametric equation of line

$$\vec{r}(t) = \begin{bmatrix} 3 \\ 4 \end{bmatrix} t = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$




WARNING: OLD NOTATION $\rightarrow (x_0, y_0)$

Point-Slope form : $y - y_0 = m(x - x_0)$

$$y - 0 = \frac{4}{3}(x - 0)$$

$$\Rightarrow \boxed{\frac{y-0}{4} = \frac{x-0}{3}}$$

symmetric equation of line

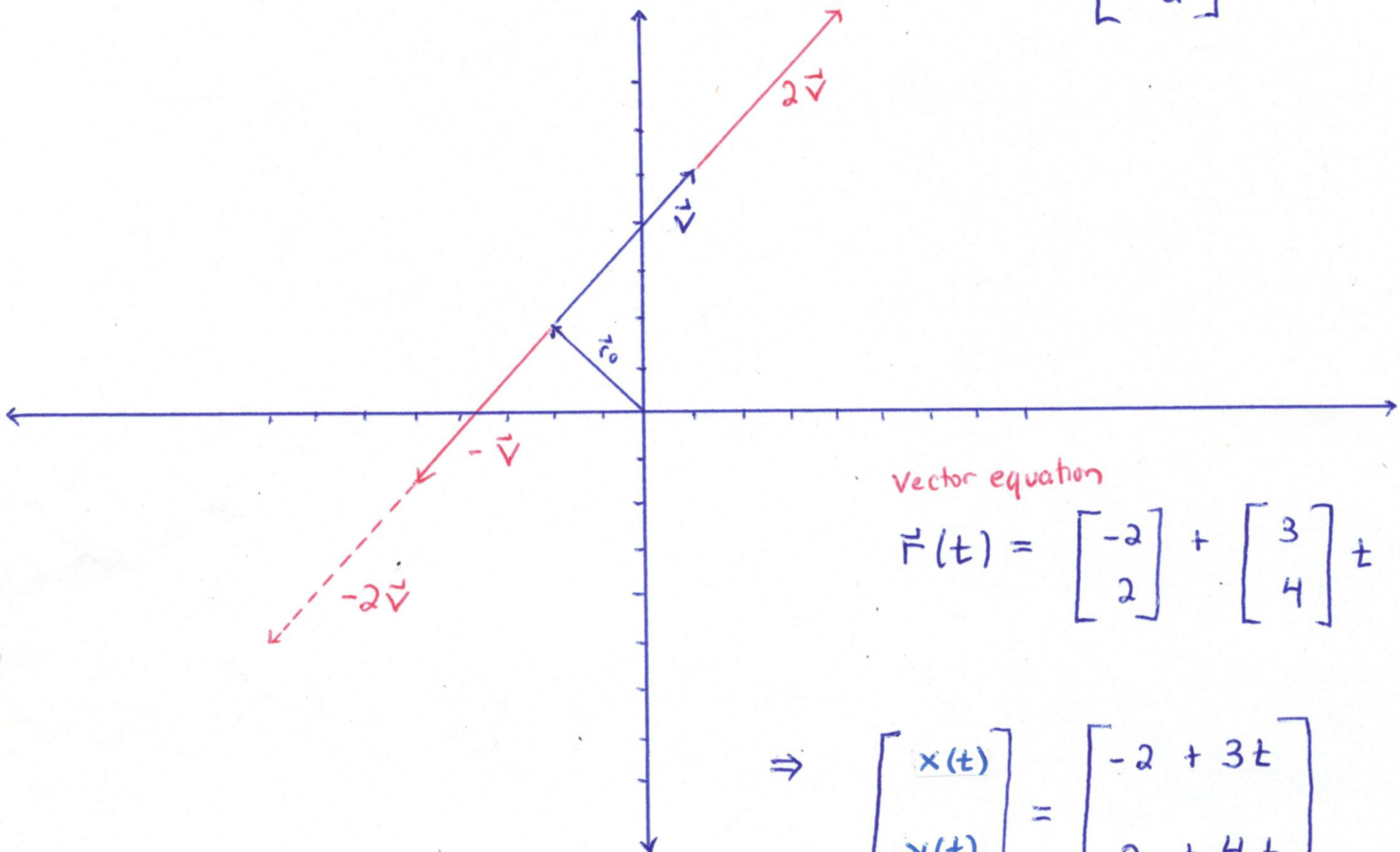
From this graph, we notice

slope $m = \frac{4}{3}$

y-intercept : $b = 0$ , $(0, 0)$

$$y = \frac{4}{3}x + 0$$

□ How might we use vector $\vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ to create a line in $\mathbb{R}^2$ through point $\vec{r}_0 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ ?




vector equation

$$\vec{r}(t) = \begin{bmatrix} -2 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} t$$

$$\Rightarrow \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -2 + 3t \\ 2 + 4t \end{bmatrix}$$

parametric equation of line

From our graph above, we see

$$\text{slope} = m = \frac{4}{3}$$

Point on graph : "$(x_0, y_0)$" $= (-2, 2)$.

$$\Rightarrow y - y_0 = m(x - x_0) : \quad y - 2 = \frac{4}{3}(x + 2)$$

$$\Rightarrow \boxed{\frac{y - 2}{4} = \frac{x - -2}{3}}$$

symmetric equation of line

□ Can you guess how we might define a general line using vector $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ through point

$$\vec{r}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

← First coordinate of $\vec{r}_0$

← second coordinate of $\vec{r}_0$

WARNING: Old notation we can also use

$$\langle x_0, y_0 \rangle = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$\vec{r}(t) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} t = \vec{r}_0 + \vec{v}\, t$$

$$\Rightarrow \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_0 + at \\ y_0 + bt \end{bmatrix}$$

← parametric equations for $x_1(t)$ and $x_2(t)$ with parameter $t$

$$\Rightarrow \quad y - y_0 = m(x - x_0), \quad m = \frac{b}{a} \begin{matrix} \leftarrow \text{rise} \\ \leftarrow \text{run} \end{matrix}$$

$$\Rightarrow \quad y - y_0 = \frac{b}{a}(x - x_0)$$

$$\Rightarrow \quad \boxed{\frac{y - y_0}{b} = \frac{x - x_0}{a}}$$

← Notice this arises when solving for $t$ in the vector form of equation for $\vec{r}(t)$

symmetric equations of Line

□ What if we wanted to generalize to the equation of a line in $\mathbb{R}^3$ using vector

$$\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

through point $\vec{r_0} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$ ← $1^{st}$ coordinate of $\vec{r_0}$ ← $2^{nd}$ coordinate of $\vec{r_0}$ ← $3^{rd}$ coordinate of $\vec{r_0}$

WARNING: OLD NOTATION
we can also use
$\langle x_0, y_0, z_0 \rangle = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$

$$\Rightarrow \boxed{\vec{r}(t) = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + \begin{bmatrix} a \\ b \\ c \end{bmatrix} t}$$

$$\Rightarrow \boxed{\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} x_0 + at \\ y_0 + bt \\ z_0 + ct \end{bmatrix}}$$

$$\Rightarrow \boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}$$

solve for $t$ in vector form of equation for $\vec{r}(t)$

WARNING: old notation $(x, y, z) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$

## Example 11.5.1 p. 800

- Let's find vector equation and parametric equation for line in the direction of $\vec{v} = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$ through point $\vec{r_0} = \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix}$

Since $\vec{r_0} = \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix}$ and we travel along $\vec{v} = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$

we know $\vec{r}(t) = \vec{r_0} + \vec{v}t = \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix} t$

$$\Rightarrow \vec{r}(t) = \begin{bmatrix} 5 + t \\ 1 + 4t \\ 3 + -2t \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

Note: We can now quickly get other points on line by choosing values of $t$:

$t = 1$ : $\vec{r}(1) = \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix}$

$t = -1$ : $\vec{r}(-1) = \begin{bmatrix} 4 \\ -3 \\ 5 \end{bmatrix}$.

## Example 11.5.2 p. 801

Let's find the parametric and symmetric equations of a line passing through points $P_0 = \begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix}$ and $P_1 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$

Solution: We know our desired line passes through $\vec{r}_0 = \begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix}$.

Moreover, since our line also passes through $P_1$, we know our line is in the direction

$$\vec{v} = \overrightarrow{P_0P_1} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix}$$

Thus, $\vec{r}(t) = \vec{r}_0 + \vec{v} \cdot t$

$$= \begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix} + \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix} t$$ vector equation

$$= \begin{bmatrix} 2 + t \\ 4 - 5t \\ -3 + 4t \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$ parametric equations

$$\Rightarrow \quad \frac{x - 2}{1} = \frac{y - 4}{-5} = \frac{z + 3}{4}$$ symmetric equations

Example 11.5.2 p. 801...

To project $\vec{r}(t)$ onto the xy-plane, we simply set the z-component of this line equal to zero.

Project $\vec{r}(t)$ onto xy-plane

$$\vec{r}_{xy}(t) = \begin{bmatrix} x(t) \\ y(t) \\ 0 \end{bmatrix} = \begin{bmatrix} 2 + t \\ 4 - 5t \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad x - 2 = \frac{y - 4}{-5}$$

$$\Rightarrow \quad -5(x-2) = y - 4$$

$$\Rightarrow \quad -5x + 10 = y - 4$$

$$\Rightarrow \quad y = -5x + 14$$

Example 11.5.2 p 801 ...

What if we wanted to find an equation for line segment from A to B only?

If we use $t=0$ : $\vec{r}(0) = \begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix} = A$

$t=1 = \vec{r}(t) = \begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix} + \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} = B$

So then segment between A & B is given by

$$\vec{r}(t) = \begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix} + \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix} t , \qquad 0 \le t \le 1$$

$$= \vec{r}_0 + [\vec{r}_1 - \vec{r}_0]\, t \qquad \text{where } \vec{r}_1 = B$$

$$= \vec{r}_0 + \vec{r}_1 t - \vec{r}_0 t$$

$$= (1-t)\vec{r}_0 + \vec{r}_1 t$$

IN General: Line segment from $\vec{r}_0$ to $\vec{r}_1$ is given by

$$\vec{r}(t) = (1-t)\vec{r}_0 + \vec{r}_1 t , \quad 0 \le t \le 1$$

LS, p12