# Math 1C: Calculus III

## Lesson 8: Limits and Continuity

**Reference:** Brigg's "Calculus: Early Transcendentals, Second Edition"
**Topics:** Section 12.3: Limits and Continuity, p. 885 - 894

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Definition. *p. 885* ***Limit of a Function of Two Variables***

A function $f(x, y)$ of two variables has a limit $L$ as $P(x, y)$ approaches a fixed point $P_0(a, b)$ if

$$|\, f(x, y) - L \,|$$

can be made arbitrarily small by forcing the point $P(x, y)$ to be sufficiently close to the point $P_0(a, b)$in the domain. If such a limit exists, we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = \lim_{P(x,y)\to P_0(a,b)} f(x, y) = L.$$

**Note:** This definition is NOT mathematically rigorous (it does not give enough detail to be used in formal proofs about limits). In order to make this definition more precise, we need to define what it means for point $P(x, y)$ to be *close to* point $P_0(a, b)$ in the domain.

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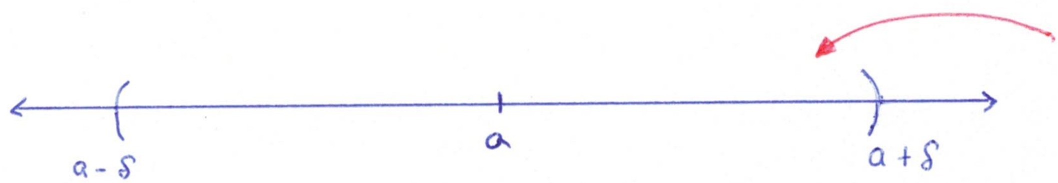
Recall from our study of Math 1A, the limit of a single variable function $f(x)$ equals $L$, written $\lim_{x\to a} f(x) = L$ iff $f(x) \to L$ as the distance between $x$ and $a$ goes to zero. We write this distance $|x - a| < \delta$ and visualize the number line




x should be inside this interval, where $\delta$ is small enough to force $f(x) \approx L$

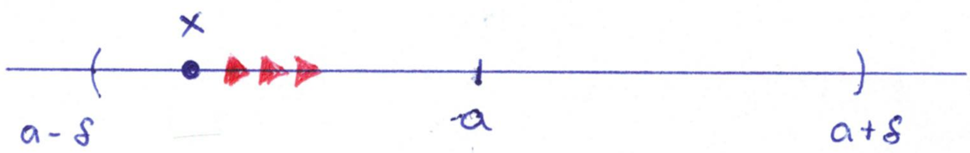
$$|x - a| < \delta \implies -\delta < x - a < \delta$$

$$\implies a - \delta < x < a + \delta$$

In this case, we said that the limiting behavior of the output of $f(x)$ had to be the same "no matter what direction" we approach $a$ in the domain.

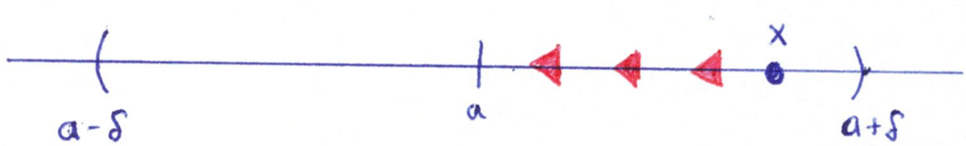
However, since $f(x)$ is a single-variable function, there are only two possible directions for $x \to a$:

- Approach from the left: $\lim_{x \to a^-} f(x)$




Here $x < a$ and
$x$ gets closer to $a$
(left-hand limit)

- Approach from the right: $\lim_{x \to a^+} f(x)$




Here, $x > a$ and
$x$ gets closer to $a$
(right-hand limit)

Recall: In order for the limit

$$\lim_{x \to a} f(x)$$

to exist, the function output must converge to the same value no matter which direction we approach $a$:

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$$

This same idea will hold for limits of multivariable functions

$$\lim_{(x,y) \to (a,b)} f(x,y)$$

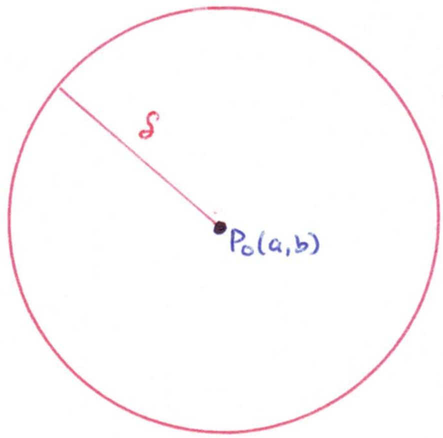
However, we notice that there are many "directions" for which $(x,y) \to (a,b)$.

For the multivariable function $f(x,y)$, we say

$\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$ iff $f(x,y) \to L$ as $(x,y)$

"gets closer" to $(a,b)$. To define the idea of

closeness in $\mathbb{R}^2$, notice




$\boxed{\text{Disk with radius } \delta:}$

$$\sqrt{(x-a)^2 + (y-b)^2} < \delta$$

edge (boundary) of disk not included since strict inequality

The same path-independent behavior should hold for multivariable limits. So, no matter what path (direction) we take to get to $(a,b)$, $f(x,y) \to L$.

**Theorem 12.1.** *p. 886* ***Limits of Constant and Linear Functions***

Suppose that $a, b$ and $c$ are real numbers. Then

1. *Constant function* $f(x,y) = c$: $$\lim_{(x,y)\to(a,b)} c = c$$

2. *Linear function* $f(x,y) = x$: $$\lim_{(x,y)\to(a,b)} f(x,y) = a$$

3. *Linear function* $f(x,y) = y$: $$\lim_{(x,y)\to(a,b)} f(x,y) = a$$

$\boxed{\text{L8, p5}}$

Theorem 12.2. *p. 887* ***Limit Laws for Multivariable Functions***

Suppose that $a, b$ and $c$ are real numbers. Let $m, n \in \mathbb{Z}$ be integers. Suppose that the limits

$$\lim_{(x,y)\to(a,b)} f(x,y) = L \qquad \text{and} \qquad \lim_{(x,y)\to(a,b)} g(x,y) = M$$

exist. Then, as long as we check these conditions, we can conclude

1. *Sum Law:* $$\lim_{(x,y)\to(a,b)} \big(f(x,y) + g(x,y)\big) = \lim_{(x,y)\to(a,b)} f(x,y) + \lim_{(x,y)\to(a,b)} g(x,y)$$

2. *Difference Law:* $$\lim_{(x,y)\to(a,b)} \big(f(x,y) - g(x,y)\big) = \lim_{(x,y)\to(a,b)} f(x,y) - \lim_{(x,y)\to(a,b)} g(x,y)$$

3. *Constant Multiple Law:* $$\lim_{(x,y)\to(a,b)} (c \cdot f(x,y)) = c \cdot \lim_{(x,y)\to(a,b)} f(x,y)$$

4. *Product Law:* $$\lim_{(x,y)\to(a,b)} (f(x,y) \cdot g(x,y)) = \left[\lim_{(x,y)\to(a,b)} f(x,y)\right] \cdot \left[\lim_{(x,y)\to(a,b)} g(x,y)\right]$$

5. *Quotient Law:* $$\lim_{(x,y)\to(a,b)} \left[\frac{f(x,y)}{g(x,y)}\right] = \frac{\left[\lim_{(x,y)\to(a,b)} f(x,y)\right]}{\left[\lim_{(x,y)\to(a,b)} g(x,y)\right]} \quad \text{if} \quad \lim_{(x,y)\to(a,b)} g(x,y) \neq 0$$

6. *Simple Power Law:* $$\lim_{(x,y)\to(a,b)} \left[\,[f(x,y)]^{\,n}\,\right] = \left[\lim_{(x,y)\to(a,b)} f(x,y)\right]^n$$

7. *General Power Law:* $$\lim_{(x,y)\to(a,b)} \left[\,[f(x,y)]^{\,m/n}\,\right] = \left[\lim_{(x,y)\to(a,b)} f(x,y)\right]^{m/n}$$

   (if $m$ and $n$ have no common factors and $n \neq 0$ and if we assume $\lim_{(x,y)\to(a,b)} f(x,y) > 0$ for $n$ even.)

L8, p6

Example 12.3.1) Evaluate $\lim\limits_{(x,y)\to(2,8)} 3x^2\cdot y + \sqrt{x\cdot y}$

Solution: First, let's take a look at the behavior of the surface as we approach (get closer to) $(a,b) = (2,8)$.

Notice: No matter which direction we approach, output behavior converges to same point.

We see this surface behaves as we expect.

We hope, then, to apply "limit laws"

$$\lim_{(x,y)\to(2,8)} 3x^2y + \sqrt{xy}$$

$$= 3\cdot \lim_{(x,y)\to(2,8)} x^2 y + \lim_{(x,y)\to(2,8)} \sqrt{xy}$$

$$= 3\cdot\left[\lim_{(x,y)\to(2,8)} x\right]^2\cdot\left[\lim_{(x,y)\to(2,8)} y\right] + \sqrt{\lim_{(x,y)\to(2,8)} x\cdot y}$$

$$= 3\cdot 2^2\cdot 8 + \sqrt{2\cdot 8} = 3\cdot 32 + 16 \neq \boxed{100}.$$

L8, p.7

**Procedure.** *p. 890* ***Two-Path Test for Nonexistence of a Limit***

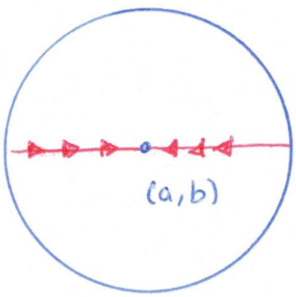
If the multivariable function $f(x, y)$ approaches two different values in the ranges as input point $(x, y)$ approaches $(a, b)$ along two different paths in the domain of $f$, then we say that the limit

$$\lim_{(x,y)\to(a,b)} f(x, y)$$

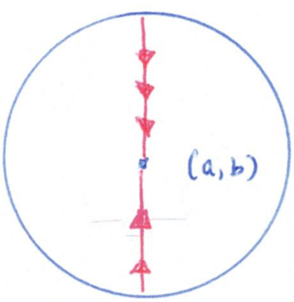
does not exist.

Remember, the idea that $f(x,y)$ gets arbitrarily close to L as $(x,y)$ gets close to $(a,b)$ must NOT depend on the path we take for the limit to exist

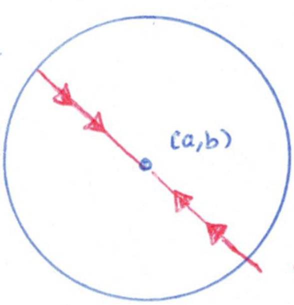
Path 1: $(x,b) \to (a,b)$
(along $y = b$ line)




Path 2: $(a,y) \to (a,b)$
(along $x = a$ line)




Path 3: $(x,y) \to (a,b)$
(along any line
$y = m(x-a) + b$
for any slope m)



Similar to
Exercise 12.3.29 ) Show that the limit
on p. 893

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

Does NOT exist.

Solution: To demonstrate

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} \quad \text{does not exist,}$$

we will find two different ~~values~~ paths $C_1$ & $C_2$ such that

Along $C_1$: $\lim\limits_{(x,y)\to(0,0)} f(x,y) = L_1$ and

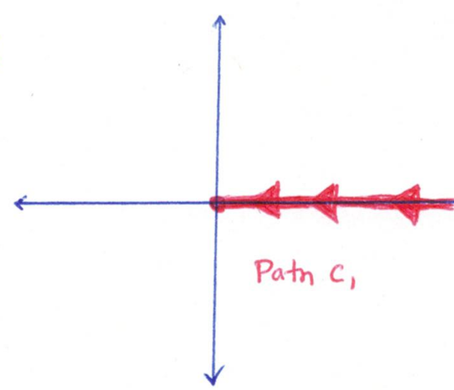
Along $C_2$: $\lim\limits_{(x,y)\to(0,0)} f(x,y) = L_2$

Path $C_1$: Let's approach $(0,0)$ along $x$-axis $(y=0)$

$$\lim_{(x,0)\to(0,0)} f(x,y) = \lim_{(x,0)\to(0,0)} \frac{x^2 - 0^2}{x^2 - 0^2}$$

$$= \lim_{x\to 0} \frac{x^2}{x^2}$$

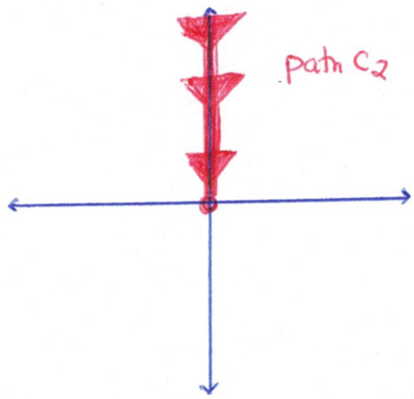
$$\boxed{= 1}$$



Similar to
Exercise 12.3.9 ....

Path $C_2$: Let's approach $(0,0)$ along y-axis $(x=0)$

$$\lim_{(0,y)\to(0,0)} f(x,y) = \lim_{(0,y)\to(0,0)} \frac{0^2 - y^2}{0^2 + y^2}$$




$$= \lim_{y\to 0} -\frac{y^2}{y^2}$$

$$= -1$$

Thus we see

Path $C_1$: $\lim_{(x,y)\to(0,0)} f(x,y) = 1$ if $(x,y)$ approaches $(0,0)$ along x-axis

Path $C_2$: $\lim_{(x,y)\to(0,0)} f(x,y) = -1$ if $(x,y)$ approachs $(0,0)$ along y-axis

Since $f(x,y)$ has two different limits along two different lines, we know

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

DOES NOT EXIST.

**Theorem.** ***Limits Obey Inequalities***

If $f(x,y) \leq g(x,y)$ for all $(x,y)$ in some open region containing $(a,b)$, except possibly at $(x,y)=(a,b)$, and the limits of $f$ and $g$ both exist as $x$ approaches $a$, then

$$\lim_{(x,y)\to(a,b)} f(x,y) \leq \lim_{(x,y)\to(a,b)} g(x,y)$$

**Theorem.** ***The Squeeze Theorem***

Suppose that $f(x,y) \leq g(x,y) \leq h(x,y)$ for all $x$ in some open region containing $(a,b)$, except possibly at $(x,y)=(a,b)$ itself. Suppose also that

$$\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(x,y)\to(a,b)} h(x,y) = L.$$

Then $\displaystyle\lim_{(x,y)\to(a,b)} g(x,y) = L$.

Similar to

Exercise 12.3.39 p. 893

Find $\displaystyle \lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$, if it exists.

Solution: Let $f(x,y) = \dfrac{3x^2y}{x^2+y^2}$

Lets try to find Behavior of $f(x,y)$ as $(x,y)\to(0,0)$

along $y=mx$:

$$f(x,y) = f(x, mx)$$

$$= \frac{3mx^3}{x^2+m^2x^2}$$

$$= \frac{3mx}{1+m^2}, \quad x \neq 0$$

$\Rightarrow f(x,y) \to 0$ as $(x,y)\to(0,0)$ along $y=mx$

Similarly, $f(x,y) = f(x, x^2)$

$$= \frac{3x^4}{x^2+x^4}$$

$$= \frac{3x^2}{1+x^2}, \quad x \neq 0$$

$\Rightarrow f(x,y) \to 0$ as $(x,y)\to(0,0)$ along $y=x^2$

We suspect $\lim_{(x,y)\to(0,0)} f(x,y) = 0$. Let's try to prove w/ squeeze thm.

To prove this, consider

$$0 \leq \left| \frac{3x^2y}{x^2+y^2} - 0 \right| = \left| \frac{3x^2y}{x^2+y^2} \right| = \frac{|3x^2y|}{|x^2+y^2|}$$

$$= \frac{3x^2 \cdot |y|}{|x^2+y^2|}$$

$$= \frac{3x^2|y|}{x^2+y^2}$$

Moreover, $x^2 \leq x^2+y^2$ since $y^2 \geq 0$ for all $y$

$$\Rightarrow \frac{x^2}{x^2+y^2} \leq 1$$

$$\Rightarrow 3|y| \cdot \frac{x^2}{x^2+y^2} \leq 3|y|$$

$$\Rightarrow 0 \leq |f(x,y)| \leq 3|y|$$

squeeze thm

$$\Rightarrow \lim_{(x,y)\to(0,0)} 0 = 0 = \lim_{(x,y)\to(0,0)} 3|y| = 0 \Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = 0 \checkmark$$

Example 12.3.4 p. 890 - 891

For $f(x,y) = \frac{xy^2}{x^2+y^4}$, does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist?

observation: $f(x,y)$ NOT "symmetric" in $x$ and $y$

$$\frac{x^{1} y^{2}}{x^{2} + y^{4}}$$

□ Can we use this information to our advantage?

Solution:

To investigate the limiting behavior of

$$f(x,y) = \frac{x \cdot y^2}{x^2 + y^4}$$

lets try moving $(x,y) \to (0,0)$ along any line through origin: $y = mx$, with $m$ the slope of our line.

Along $y=mx$:

$$f(x,y) = f(x, mx)$$

$$= \frac{x \cdot [mx]^2}{x^2 + [mx]^4}$$

$$= \frac{m^2 x^3}{x^2 + m^4 x^4}$$

$$= \frac{m^2 x}{1 + m^4 x^2}.$$

$\Rightarrow$ As $(x,y) \to (0,0)$ along $y = mx$, we have

$$\underbrace{f(x,y)}_{} \to 0 \quad \text{as} \quad (x,y) \to (0,0) \text{ along } y = mx.$$

has same limiting value along every nonvertical line through origin

Example 12.3.4 p. 890 – 891 ...

□ Just because $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along $y = mx$, can we conclude $\lim_{(x,y)\to(0,0)} f(x,y)$ exists?

* restate definition of $\lim_{(x,y)\to(a,b)} f(x,y)$

Let's consider $(x,y) \to (0,0)$ along parabola $x = y^2$

$$\Rightarrow \quad f(x,y) = f(y^2, y)$$

$$= \frac{y^2 y^2}{[y^2]^2 + y^4}$$

$$= \frac{y^4}{2y^4}$$

$$= \frac{1}{2} \quad \text{for } y \neq 0$$

$\Rightarrow \quad f(x,y) \to \frac{1}{2}$ as $(x,y) \to (0,0)$ along $x = y^2$.

Since $f(x,y)$ has two different limits along two different paths approaching $(0,0)$, we know

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$$

DOES NOT exist.

**Guidelines for Finding Non-Obvious Limits of Multivariable Functions**

Here are some suggestions when testing to find a limit of a multivariable function $f(x, y)$:

1. Try to identify if $f(x, y)$ is continuous at the point $(a, b)$. If so, you can use direct substitution to evaluate the limit since

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

2. Try finding $f(x, y) \longrightarrow M$ as $(x, y) \longrightarrow (a, b)$ along the a horizontal line with constant $y = b$ value. In other words, find the behavior of the output of $f(x, y)$ as

$$(x, b) \longrightarrow (a, b)$$

3. Try finding $f(x, y) \longrightarrow M$ as $(x, y) \longrightarrow (a, b)$ along the a vertical line with constant $x = a$ value. In other words, find the behavior of the output of $f(x, y)$ as

$$(a, y) \longrightarrow (a, b)$$

4. Look carefully at the algebraic expression that defines function $f(x, y)$. Try to find a creative path for $(x, y)$ in the domain that approaches $(a, b)$ that will result in a single-variable limit that might be instructive. For example, you might try

   A. Lines $y = mx$ with $(x, mx) \to (0, 0)$

   B. Parabola $y = x^2$ with $(x, x^2) \to (0, 0)$

   C. Parabola $x = y^2$ with $(y^2, y) \to (0, 0)$

Note: If you have reason to believe that a limit exists, you might also try the squeeze thm as we showed in our example.

**Definition.** *p. 888* ***Continuity for Multivariable Functions***

The multivariable function $f(x,y)$ is continuous at the point $(a,b)$ provided that the following three conditions hold:

1. The output value $f(a,b)$ is defined (i.e. the point $(a,b)$ is in the domain of function $f$)
2. The limit $\lim_{(x,y)\to(a,b)} f(x,y)$ exists
3. $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

Similar to Math 1A in that limits of continuous functions can be evaluated by direct substitution

Similar to Example 12.4.4 p. 890)

Determine the points at which the following function is continuous

$$f(x,y) = \begin{cases} \dfrac{3x^2y}{x^4+y^2} & \text{if } (x,y)\neq(0,0) \\ (0,0) & \text{if } (x,y)=(0,0) \end{cases}$$

If $(x,y)\neq(0,0)$, we see $f(x,y)$ is a rational function so it is continuous on all points in domain (in this case the domain is $\mathbb{R}^2 - \{(0,0)\}$.

similar to Example 12.4.4 p. 890 ...

Along path $x = my$ with $(x, y) \neq (0,0)$

$$f(x,y) = \frac{3\,[my]^2 \cdot y}{[my]^4 + y^2}$$

$$= \frac{3m^2 \cdot y^3}{m^4 y^4 + y^2}$$

$$= \frac{3m^2\, y}{1 + m^4 y^2}$$

As $(x,y) \to (0,0)$ along $(my, y)$, we see $f(x,y) \to 0$

However, as before we try parabolic path $y = x^2$

$$f(x,y) = \frac{3 \cdot x^2 \cdot x^2}{x^4 + [x^2]^2} = \frac{3\, x^4}{2\, x^4} = \frac{3}{2}$$

$$\Rightarrow \lim_{(x,x^2) \to (0,0)} f(x,y) = \frac{3}{2} \neq 0.$$

Thus, $f(x,y)$ is discontinuous at $(0,0)$.

**Theorem 12.3.** *p. 891* ***Continuity of Composite Functions***

If $u = g(x,y)$ is continuous at $(a,b)$ and $z = f(u)$ is continuous at $g(a,b)$, then the composite function

$$z = f(g(x,y))$$

is continuous at $(a,b)$.

**Theorem.** ***Direct Substitution Property for Polynomials***

Suppose $n \in \mathbb{N}$. If $f(x,y) = c_{mn}x^m y^n + \cdots + c_{10}x + c_{01}y + c_{00}$ is a polynomial with $c_{ik} \in \mathbb{R}$ for all $i, k$ and $a$ is in the domain of $f$, then

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b) = c_{mn}a^m b^n + \cdots + c_{10}a + c_{01}b + c_{00}$$

**Theorem.** ***Direct Substitution Property for Rational Functions***

Define polynomials $f(x,y)$ and $g(x,y)$. Suppose that $(a,b)$ is in the domain of $f$ and $g$ with $g(a,b) \neq 0$, then

$$\lim_{(x,y)\to(a,b)} \left[\frac{f(x,y)}{g(x,y)}\right] = \left[\frac{f(a,b)}{g(a,b)}\right]$$

**Theorem.** ***Limits at Removable Discontinuities***

If $f(x,y) = g(x,y)$ when $x \neq a, y \neq b$, then

$$\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(x,y)\to(a,b)} g(x,y)$$

provided the limits exist.