Lesson 8: Multivariable Limits and Continuity Handout
Reference: Brigg's "Calculus: Early Transcendentals, Second Edition"
Topics: Section 12.3: Limits and Continuity, p. 885 - 894

Definition. *p. 885* ***Limit of a Function of Two Variables***

A function $f(x, y)$ of two variables has a limit L as $P(x, y)$ approaches a fixed point $P_0(a, b)$ if

$$|\, f(x, y) - L \,|$$

can be made arbitrarily small by forcing the point $P(x, y)$ to be sufficiently close to the point $P_0(a, b)$in the domain. If such a limit exists, we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = \lim_{P(x,y)\to P_0(a,b)} f(x, y) = L.$$

Note: This definition is NOT mathematically rigorous (it does not give enough detail to be used in formal proofs about limits). In order to make this definition more precise, we need to define what it means for point $P(x, y)$ to be *close to* point $P_0(a, b)$ in the domain.

Definition. *p. 886* ***Precise Definition of a Limit of a Function of Two Variables***

A function $f(x, y)$ of two variables has a limit L as point $P(x, y)$ in the domain of f approaches a fixed point $P_0(a, b)$, written as

$$\lim_{(x,y)\to(a,b)} f(x, y) = \lim_{P(x,y)\to P_0(a,b)} f(x, y) = L,$$

if, for any $\epsilon > 0$, there exists a $\delta > 0$ such that if

$$0 < \left\| \overrightarrow{PP_0} \right\|_2 = \sqrt{(x-a)^2 + (y-b)^2} < \delta$$

then

$$|\, f(x, y) - L \,| < \epsilon$$

Note: The condition that

$$\left\| \overrightarrow{PP_0} \right\|_2 < \delta$$

means that the distance between domain point $P(x, y)$ and $P_0(a, b)$ is less than δ as P approaches P_0 from *all* possible directions. Therefor, the limit exists only if $f(x, y)$ approaches L as P approaches P_0 *along all possible paths* in the domain of f. As shown in the examples discussed in class, this interpretation is critical to determine if a limit of a multivariable function exists.

Theorem 12.1. *p. 886* ***Limits of Constant and Linear Functions***

Suppose that a, b and c are real numbers. Then

1. *Constant function* $f(x,y) = c$*:* $$\lim_{(x,y)\to(a,b)} c = c$$

2. *Linear function* $f(x,y) = x$*:* $$\lim_{(x,y)\to(a,b)} f(x,y) = a$$

3. *Linear function* $f(x,y) = y$*:* $$\lim_{(x,y)\to(a,b)} f(x,y) = a$$

Theorem 12.2. *p. 887* ***Limit Laws for Multivariable Functions***

Suppose that a, b and c are real numbers. Let $m, n \in \mathbb{Z}$ be integers. Suppose that the limits

$$\lim_{(x,y)\to(a,b)} f(x,y) = L \qquad \text{and} \qquad \lim_{(x,y)\to(a,b)} g(x,y) = M$$

exist. Then, as long as we check these conditions, we can conclude

1. *Sum Law:* $$\lim_{(x,y)\to(a,b)} \big(f(x,y) + g(x,y)\big) = \lim_{(x,y)\to(a,b)} f(x,y) + \lim_{(x,y)\to(a,b)} g(x,y)$$

2. *Difference Law:* $$\lim_{(x,y)\to(a,b)} \big(f(x,y) - g(x,y)\big) = \lim_{(x,y)\to(a,b)} f(x,y) - \lim_{(x,y)\to(a,b)} g(x,y)$$

3. *Constant Multiple Law:* $$\lim_{(x,y)\to(a,b)} (c \cdot f(x,y)) = c \cdot \lim_{(x,y)\to(a,b)} f(x,y)$$

4. *Product Law:* $$\lim_{(x,y)\to(a,b)} (f(x,y) \cdot g(x,y)) = \left[\lim_{(x,y)\to(a,b)} f(x,y)\right] \cdot \left[\lim_{(x,y)\to(a,b)} g(x,y)\right]$$

5. *Quotient Law:* $$\lim_{(x,y)\to(a,b)} \left[\frac{f(x,y)}{g(x,y)}\right] = \frac{\left[\lim_{(x,y)\to(a,b)} f(x,y)\right]}{\left[\lim_{(x,y)\to(a,b)} g(x,y)\right]} \text{ if } \lim_{(x,y)\to(a,b)} g(x,y) \neq 0$$

6. *Simple Power Law:* $$\lim_{(x,y)\to(a,b)} \Big[\, [f(x,y)]^{\,n} \,\Big] = \Big[\lim_{(x,y)\to(a,b)} f(x,y) \Big]^n$$

7. *General Power Law:* $$\lim_{(x,y)\to(a,b)} \Big[\, [f(x,y)]^{\,m/n} \,\Big] = \Big[\lim_{(x,y)\to(a,b)} f(x,y) \Big]^{m/n}$$

   (if m and n have no common factors and $n \neq 0$ and if we assume $\lim_{(x,y)\to(a,b)} f(x,y) > 0$ for n even.)

Theorem. ***Limits Obey Inequalities***

If $f(x,y) \leq g(x,y)$ for all (x,y) in some open region containing (a,b), except possibly at $(x,y)=(a,b)$, and the limits of f and g both exist as x approaches a, then

$$\lim_{(x,y)\to(a,b)} f(x,y) \leq \lim_{(x,y)\to(a,b)} g(x,y)$$

Theorem. ***The Squeeze Theorem***

Suppose that $f(x,y) \leq g(x,y) \leq h(x,y)$ for all x in some open region containing (a,b), except possibly at $(x,y)=(a,b)$ itself. Suppose also that

$$\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(x,y)\to(a,b)} h(x,y) = L.$$

Then $\lim_{(x,y)\to(a,b)} g(x,y) = L$.

Definition. *p. 888* ***Interior Point***

Let R be a region in $\mathbb{R}^2$. An **interior point** P of R lies "entirely within" R. In other words, we can always find a disk, whose center is at point P, that contains only points in R.

Definition. *p. 888* ***Boundary Point***

Let R be a region in $\mathbb{R}^2$. A **boundary point** P of R lies "on the edge" of R. In other words, very disk, whose center is at point P, contains at least one point in R and at least one point not in R.

Definition. *p. 888* ***Open Set***

A region R in $\mathbb{R}^2$ is **open** if it consists entirely of interior point.

Definition. *p. 888* ***Closed Set***

A region R in $\mathbb{R}^2$ is **closed** if it contains all of its boundary points.

Procedure. *p. 890* ***Two-Path Test for Nonexistence of a Limit***

If the multivariable function $f(x, y)$ approaches two different values in the ranges as input point (x, y) approaches (a, b) along two different paths in the domain of f, then we say that the limit

$$\lim_{(x,y)\to(a,b)} f(x, y)$$

does not exist.

Definition. *p. 888* ***Continuity for Multivariable Functions***

The multivariable function $f(x, y)$ is continuous at the point (a, b) provided that the following three conditions hold:

1. The output value $f(a, b)$ is defined (i.e. the point (a, b) is in the domain of function f)
2. The limit $\lim\limits_{(x,y)\to(a,b)} f(x, y)$ exists
3. $\lim\limits_{(x,y)\to(a,b)} f(x, y) = f(a, b)$

Theorem 12.3. *p. 891* ***Continuity of Composite Functions***

If $u = g(x, y)$ is continuous at (a, b) and $z = f(u)$ is continuous at $g(a, b)$, then the composite function

$$z = f(g(x, y))$$

is continuous at (a, b).

Theorem. ***Direct Substitution Property for Polynomials***

Suppose $n \in \mathbb{N}$. If $f(x, y) = c_{mn}x^m y^n + \cdots + c_{10}x + c_{01}y + c_{00}$ is a polynomial with $c_{ik} \in \mathbb{R}$ for all i, k and a is in the domain of f, then

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b) = c_{mn}a^m b^n + \cdots + c_{10}a + c_{01}b + c_{00}$$

Theorem. ***Direct Substitution Property for Rational Functions***

Define polynomials $f(x, y)$ and $g(x, y)$. Suppose that (a, b) is in the domain of f and g with $g(a, b) \neq 0$, then

$$\lim_{(x,y)\to(a,b)} \left[\frac{f(x,y)}{g(x,y)}\right] = \left[\frac{f(a,b)}{g(a,b)}\right]$$

Theorem. ***Limits at Removable Discontinuities***

If $f(x, y) = g(x, y)$ when $x \neq a, y \neq b$, then $\displaystyle\lim_{(x,y)\to(a,b)} f(x, y) = \lim_{(x,y)\to(a,b)} g(x, y)$ provided the limits exist.

Guidelines for Finding Non-Obvious Limits of Multivariable Functions

Here are some suggestions when testing to find a limit of a multivariable function $f(x, y)$:

1. Try to identify if $f(x, y)$ is continuous at the point (a, b). If so, you can use direct substitution to evaluate the limit since

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

2. Try finding $f(x, y) \longrightarrow M$ as $(x, y) \longrightarrow (a, b)$ along the a horizontal line with constant $y = b$ value. In other words, find the behavior of the output of $f(x, y)$ as

$$(x, b) \longrightarrow (a, b)$$

3. Try finding $f(x, y) \longrightarrow M$ as $(x, y) \longrightarrow (a, b)$ along the a vertical line with constant $x = a$ value. In other words, find the behavior of the output of $f(x, y)$ as

$$(a, y) \longrightarrow (a, b)$$

4. Look carefully at the algebraic expression that defines function $f(x, y)$. Try to find a creative path for (x, y) in the domain that approaches (a, b) that will result in a single-variable limit that might be instructive. For example, you might try

   A. Lines $y = mx$ with $(x, mx) \to (0, 0)$

   B. Parabola $y = x^2$ with $(x, x^2) \to (0, 0)$

   C. Parabola $x = y^2$ with $(y^2, y) \to (0, 0)$