Lesson 9: Partial Derivatives

Recall the major themes we've discussed in Math 1C

Single Variable Calculus: Math 1A/1B

- Focuses on studying special operations (differentiation, integration) on single-variable functions

$$\underbrace{f: D}_{\text{domain}} \longrightarrow \underbrace{\mathbb{R}}_{\text{codomain}}, \qquad D \subseteq \mathbb{R}$$

Here we see that $f(x)$ requires a <u>single</u> real-valued input: hence f is a single-variable function

Differential Calculus: Math 1A

Studies the "forward" problem of taking ordinary derivatives

derivative operation defined via limits in math 1A $\longrightarrow$

$$\frac{d}{dx}\Big[\underset{\substack{\uparrow \\ \text{Given}}}{F(x)}\Big] = \underset{\substack{\uparrow \\ \text{unknown}}}{f(x)} = F'(x)$$

Multivariable Calculus : Math 1C/1D

- Focuses on studying special operations (partial differentiation, partial integrals) on multivarible functions

$$f : \underbrace{D}_{\text{domain}} \longrightarrow \underbrace{\mathbb{R}}_{\text{Codomain}} , \qquad D \subseteq \mathbb{R}^2 \quad \text{or} \quad D \subseteq \mathbb{R}^3$$

Here we see that f(x) requires two ($\mathbb{R}^2$) or three ($\mathbb{R}^3$) real-valued inputs: hence f is multivarrable

Differential Calculus for Multivariable Functions : Math 1C

Studies the "forward" problem of taking partial derivatives

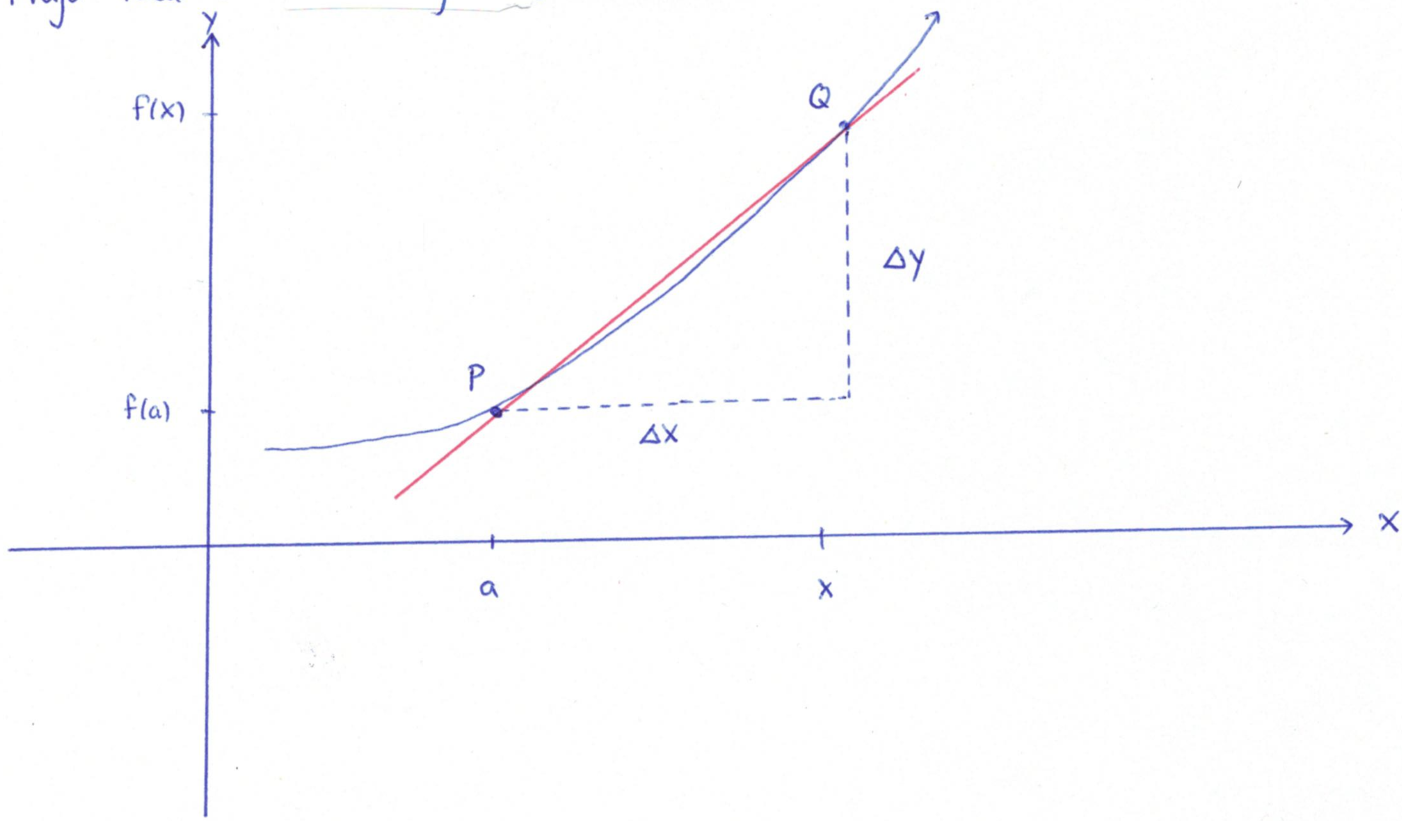
$$\frac{\partial}{\partial x_i}\left[F(x_1, x_2)\right] = f(x_1, x_2) = F_{x_i}(x_1, x_2)$$

Given

unknown

partial derivative operation defined via limits in Math 1C

Major Ideas of Ordinary Differentiation



Slope of Secant line through Point PQ

$$m_{PQ} = \frac{\Delta y}{\Delta x} = \frac{f(x) - f(a)}{x - a}$$

Also known as the average rate of change between P and Q

To find the slope of the tangent line at point P, we let x get infinitely close to a using limit operation

$$f'(a) = \lim_{x \to a} \left[\frac{f(x) - f(a)}{x - a} \right]$$

- limit definition of derivative using slope notation

$$= \lim_{h \to 0} \left[\frac{f(a+h) - f(a)}{h} \right]$$

- limit definition of derivative using derivative notation

where $h = x - a \Rightarrow x = a + h$

Let $f(x,y)$ be a multivariable function.

If we fix $y=b$ constant, then $f(x,y) = f(x,b)$ is a single variable function.

derivative at a point

The partial derivative of f w/ respect to x at (a,b) is

$$f_x(a,b) = g'(a) \qquad \text{where} \qquad g(x) = f(x,b)$$

limit in first coordinate

2nd coordinate held constant

$$= \lim_{h \to 0} \frac{f(a+h,\, b) - f(a,\, b)}{h}$$

$$f_x(x,y) = f_x$$

$$= \frac{\partial}{\partial x}\left[f(x,y)\right]$$

$$= \frac{\partial f}{\partial x}$$

$$= D_x f$$

For $f(x,y) = f(x_1, x_2)$, we can also find the derivative of f w/ respect to y at (a,b) by fixing $x = a$ and considering

1st coordinate held constant

limit in second coordinate

$$f_y(a,b) = \lim_{h \to 0} \frac{f(a, b+h) - f(a,b)}{h}$$

$$f_y(x,y) = f_y$$

$$= \frac{\partial}{\partial y} f(x,y)$$

$$= \frac{\partial f}{\partial y}$$

$$= D_y f$$

$$= D_2 f$$

Partial Derivatives as functions

If $f(x,y) = f(x_1, x_2)$ is a function of two variables, ~~if~~ its partial derivative functions f_x and f_y are defined by

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Rules for finding partial derivatives

- To find f_x, regard y as a "constant" and differentiate $f(x,y)$ w/r to x
- To find f_y, regard x as a "constant" and differentiate $f(x,y)$ w/ respect to y.

Similar to Example 12.4.2 p. 898) Let $f(x,y) = x^3 + x^2y^3 - 2y^2$. Find $f_x(2,1)$ & $f_y(2,1)$

Solution: First let's find

$$f_x(x,y) = \frac{\partial}{\partial x}\left[x^3 + x^2y^3 - 2y^2\right]$$

$$= 3x^2 + 2xy^3 - 0$$

$$= 3x^2 + 2xy^3$$

$$\Rightarrow f_x(2,1) = 12 + 4 = \boxed{16.}$$

Next, we find

$$f_y(x,y) = \frac{\partial}{\partial y}\left[x^3 + x^2y^3 - 2y^2\right]$$

$$= 0 + 3x^2y^2 - 4y$$

$$= 3x^2y^2 - 4y$$

$$\Rightarrow f_y(2,1) = 12 - 4 = \boxed{8}.$$

Similar to

Example 12.4.3 p. 898) Let $f(x,y) = \sin\left(\frac{x}{1+y}\right)$. Find f_x & f_y.

Solution: Lets start with our partial derivative with respect to x:

$$f_x(x,y) = \frac{\partial}{\partial x}\left[f(x,y)\right]$$

$$= \frac{\partial}{\partial x}\left[\sin\left(\frac{x}{1+y}\right)\right]$$

$$= \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial x}\left[\frac{x}{1+y}\right]$$

$$= \boxed{\frac{1}{1+y} \cdot \cos\left(\frac{x}{1+y}\right)}$$

Now, let's move onto our partial derivative w/r to y:

$$f_y(x,y) = \frac{\partial}{\partial y}\left[\sin\left(\frac{x}{1+y}\right)\right]$$

$$= \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial y}\left[x \cdot (1+y)^{-1}\right]$$

$$= \boxed{\frac{-x}{(1+y)^2} \cdot \cos\left(\frac{x}{1+y}\right)}$$

Higher Order Partial Derivatives

Notice that if $f: D \to \mathbb{R}$ where $D \subseteq \mathbb{R}^2$, the partial derivatives f_x & f_y are functions of two variables, so we can consider

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial^2 f}{\partial x^2}$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial^2 f}{\partial y\,\partial x}$$

$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial^2 f}{\partial x\,\partial y}$$

$$(f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial^2 f}{\partial y^2}$$

Similar to

Example 12.4.4 p.899

Let $f(x,y) = x^3 + x^2 y^3 - 2y^2$. Find f_x, f_y, f_{xx}, f_{xy}, f_{yx}, f_{yy}

$$f_x = \frac{\partial}{\partial x}\left[x^3 + x^2 y^3 - 2y^2\right] = 3x^2 + 2xy^3$$

$$f_y = \frac{\partial}{\partial y}\left[x^3 + x^2 y^3 - 2y^2\right] = 3y^2 x^2 - 4y$$

similar

Example 12.4.4 p 899 ...

$$f_{xx} = \frac{\partial}{\partial x}\left[f_x\right]$$

$$= \frac{\partial}{\partial x}\left[3x^2 + 2xy^3\right]$$

$$= 6x + 2y^3$$

$$f_{yy} = \frac{\partial}{\partial y}\left[f_y\right] = \frac{\partial}{\partial y}\left[3y^2x^2 - 4y\right]$$

$$= 6yx^2 - 4$$

$$f_{yx} = \frac{\partial}{\partial x}\left[f_y\right] = \frac{\partial}{\partial x}\left[3y^2x^2 - 4y\right]$$

$$= 6y^2x$$

Notice! These are equal

$$f_{xy} = \frac{\partial}{\partial y}\left[f_x\right] = \frac{\partial}{\partial y}\left[3x^2 + 2xy^3\right]$$

$$= 6xy^2$$

Clairaut's Theorem

Suppose f is defined on a disk D that contains the point (a,b). If functions f_{xy} and f_{yx} are both continuous on D

then $$f_{xy}(a,b) = f_{yx}(a,b)$$

Partial Derivatives of functions of more than two variables

Suppose $f(x,y,z)$ is $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}^3$. Then

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y, z) - f(x,y,z)}{h}$$

can be found by regarding y, z as constants and differentiating with respect to x.

Similar to Example: 12.4.5 p. 900) Let $f(x,y,z) = e^{x \cdot y} \cdot \ln(z)$

Then $f_x = \frac{\partial}{\partial x}\left[e^{xy} \cdot \ln(z)\right]$

$$= \boxed{y\, e^{xy} \cdot \ln(z)}$$

$$f_y = \frac{\partial}{\partial y}\left[e^{xy} \ln(z)\right] = \boxed{x\, e^{xy} \cdot \ln(z)}$$

$$f_z = \frac{\partial}{\partial z}\left[e^{xy} \ln(z)\right] = \boxed{\frac{e^{xy}}{z}}$$

Example:

Find u_x, u_y, u_z for

$$u(x,y,z) = 3x \cdot y \cdot \sin^{-1}(y \cdot z)$$

Solution: Here we apply are rules for differentiation.

$$u_x = \frac{\partial}{\partial x}\left[u(x,y,z)\right] = \frac{\partial}{\partial x}\left[3 \cdot x \cdot y \ \arcsin(y \cdot z)\right] = 3 \cdot y \cdot \arcsin(y \cdot z)$$

$$u_y = \frac{\partial}{\partial y}\left[u(x,y,z)\right] = \frac{\partial}{\partial y}\left[3 \cdot x \cdot y \cdot \arcsin(y \cdot z)\right]$$

$$= 3x \arcsin(y \cdot z) + 3xy \cdot \frac{z}{\sqrt{1 - y \cdot z}}$$

$$u_z = \frac{\partial}{\partial z}\left[u(x,y,z)\right] = \frac{\partial}{\partial z}\left[3xy \ \arcsin(y \cdot z)\right]$$

$$= 3 \cdot x \cdot y \cdot \frac{y}{\sqrt{1 - y \cdot z}}$$

Example. Lets consider the relation

$$x^3 + y^3 + z^3 + 6xyz = 1$$

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ using implicit differentiation.

Solution: $\frac{\partial}{\partial x}\left[x^3 + y^3 + z^3 + 6xyz\right] = \frac{\partial}{\partial x}[1]$

□ everywhere we see z, we think of z as a function of x: $z = z(x)$.

$$\Rightarrow 3x^2 + 3z^2 \frac{\partial z}{\partial x} + 6yz + 6xy\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \frac{\partial z}{\partial x}\left[3z^2 + 6xy\right] = -3x^2 - 6yz$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial x} = -\frac{(x^2 + 2yz)}{z^2 + 2xy}} \checkmark$$

$$\frac{\partial}{\partial y}\left[x^3 + y^3 + z^3 + 6xyz\right] = \frac{\partial}{\partial y}[1]$$

$$\Rightarrow 3y^2 + 3z^2\frac{\partial z}{\partial y} + 6xz + 6xy\frac{\partial z}{\partial y} = 0$$

$$\Rightarrow \frac{\partial z}{\partial y}\left(3z^2 + 6xy\right) = -\left(3y^2 + 6xz\right)$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial y} = -\frac{(y^2 + 2xz)}{z^2 + 2xy}} \checkmark$$

Theorem: if the partial derivatives f_x and f_y exist near (a,b) and are continuous at (a,b), then f is differentiable at (a,b).

Definition: if $z = f(x,y)$, then f is differentiable if Δz can be expressed as

$$\Delta z = f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.