Lesson 9: Ordinary Derivatives Review Handout
Reference: Brigg's "Calculus: Early Transcendentals, Second Edition"
Topics: Chapter 3: Derivatives, p. 126 - 235

Definition. ***Limit Definition of Derivatives at a point for Single-Variable Functions***

Let $f : D \to \mathbb{R}$ be a single variable function, where $D = \mathbf{Dom}(f) \subseteq \mathbb{R}$. Then, the **derivative of a function** f **at a number** $a \in D$, denoted $f'(a)$ is given by

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \qquad \text{Limit definition of the at a point in slope notation.}$$

$$= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \qquad \text{Limit definition of the at a point in derivative notation.}$$

The tangent line to the single variable function $y = f(x)$ at point $(a, f(a))$ is the line through $(a, f(a))$ whose slope is equal to $f'(a)$, the derivative of $f(x)$ at $x = a$. We call the derivative $f'(a)$ the instantaneous rate of change of $y = f(x)$ with respect to x at $x = a$.

Definition. ***Limit Definition of Derivatives as a function for Single-Variable Functions***

Let $f : D \to \mathbb{R}$ be a single variable function, where $D = \mathbf{Dom}(f) \subseteq \mathbb{R}$. Then, the **derivative of a function** f **at any number** $x \in D$, denoted $f'(x)$ is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad \text{Limit definition of the derivative in derivative notation.}$$

The function $f'(x)$ is a function of variable x, known as the derivative of f with respect to x.

Theorem. ***Differentiability Implies Continuity***

If f is differentiable at a, then f is continuous at a.

Theorem. ***Increasing Test and Decreasing Test***

Increasing Test: If $f'(x) > 0$ on interval $I \subset D$, then f is increasing on this interval.
Decreasing Test: If $f'(x) < 0$ on interval $I \subset D$, then f is decreasing on this interval.

Theorem. ***Concave Up Test and Concave Down Test***

Concave Up Test: If $f''(x) > 0$ on interval $I \subset D$, then f is concave up on this interval.
Concave Down Test: If $f''(x) < 0$ on interval $I \subset D$, then f is concave down on this interval.

Theorem. ***Derivative Rules for Algebraic Combinations of Functions***

Suppose that a are real numbers and f and g are differentiable functions. Further, for the chain rule suppose that f is differentiable at $g(x)$. Then, as long as we check these conditions, we can conclude

1. *Constant Multiple Rule:* $$\frac{d}{dx}\left[c \cdot f(x)\right] = c \cdot \frac{d}{dx}\left[f(x)\right]$$

2. *Sum/Difference Rule:* $$\frac{d}{dx}\left[f(x) \pm g(x)\right] = \frac{d}{dx}\left[f(x)\right] \pm \frac{d}{dx}\left[g(x)\right]$$

3. *Product Rule:* $$\frac{d}{dx}\left[f(x) \cdot g(x)\right] = f(x) \cdot \frac{d}{dx}\left[g(x)\right] + g(x) \cdot \frac{d}{dx}\left[f(x)\right]$$

4. *Quotient Rule:* $$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x) \cdot \frac{d}{dx}\left[f(x)\right] - f(x) \cdot \frac{d}{dx}\left[g(x)\right]}{\left[g(x)\right]^2}$$

5. *Chain Rule:* $$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot f'(x)$$

Theorem. ***Derivative Rules for Power, Exponential and Logarithmic Functions***

Suppose that a, c are real numbers and $n \in \mathbb{R}$. Then, as long as we check these conditions, we can conclude

Constant Rule: $\frac{d}{dx}[c] = 0$

Simple Power Rule: If $n \in \mathbb{N}$, then $\frac{d}{dx}[x^n] = n \cdot x^{n-1}$

General Power Rule: If $n \in \mathbb{R}$, then $\frac{d}{dx}[x^n] = n \cdot x^{n-1}$

Derivative of Exponent Rule: $\frac{d}{dx}[a^x] = a^x \cdot \ln(a)$

Derivative of Natural Exponent Rule: $\frac{d}{dx}[e^x] = e^x$

Derivative of $\ln(|x|)$*:* $\frac{d}{dx}\left[\ln\left(|x|\right)\right] = \frac{1}{x}$

Derivative of $\log_a(x)$*:* $\frac{d}{dx}\left[\log_a(x)\right] = \frac{1}{x \cdot \ln(a)}$

Theorem. ***Derivative Rules for Trigonometric Functions***

Suppose that a, c are real numbers and $n \in \mathbb{R}$. Then, as long as we check these conditions, we can conclude

Derivative of $\sin(x)$ *Rule:* $$\frac{d}{dx}[\sin(x)] = \cos(x)$$

Derivative of $\cos(x)$ *Rule:* $$\frac{d}{dx}[\cos(x)] = -\sin(x)$$

Derivative of $\tan(x)$ *Rule:* $$\frac{d}{dx}[\tan(x)] = \sec^2(x)$$

Derivative of $\csc(x)$ *Rule:* $$\frac{d}{dx}[\csc(x)] = -\csc(x) \cdot \cot(x)$$

Derivative of $\sec(x)$ *Rule:* $$\frac{d}{dx}[\sec(x)] = \sec(x) \cdot \tan(x)$$

Derivative of $\cot(x)$ *Rule:* $$\frac{d}{dx}[\cot(x)] = -\csc^2(x)$$

Theorem. ***Derivative Rules for Trigonometric Functions***

Suppose that a, c are real numbers and $n \in \mathbb{R}$. Then, as long as we check these conditions, we can conclude

Derivative of $\arcsin(x)$ *Rule:* $$\frac{d}{dx}[\arcsin(x)] = \frac{1}{\sqrt{1-x^2}}$$

Derivative of $\arccos(x)$ *Rule:* $$\frac{d}{dx}[\arccos(x)] = \frac{-1}{\sqrt{1-x^2}}$$

Derivative of $\arctan(x)$ *Rule:* $$\frac{d}{dx}[\arctan(x)] = \frac{1}{1+x^2}$$

Derivative of $\text{arccsc}(x)$ *Rule:* $$\frac{d}{dx}[\sec(x)] = -\frac{1}{x\sqrt{x^2-1}}$$

Derivative of $\text{arcsec}(x)$ *Rule:* $$\frac{d}{dx}[\sec(x)] = \frac{1}{x\sqrt{x^2-1}}$$

Derivative of $\text{arccot}(x)$ *Rule:* $$\frac{d}{dx}[\cot(x)] = -\frac{1}{1+x^2}$$

Procedure. ***Implicit Differentiation***

Implicit differentiation is a technique used to find $y' = \frac{dy}{dx}$ when we are considering an equation involving variables x and y such that y cannot be written explicitly in terms of x. In this case, we apply implicit differentiation using a number of steps:

1. Take derivative of both sides of implicit equation with respect to x
2. Assume that "output" variable y is an explicit function of "input" variable: x: i.e. assume $y = y(x)$
3. Apply chain rule and other rules of differentiation to both sides of equation. For the derivative of the inner function, introduce algebraic symbol representing unknown derivative: $y' = \frac{dy}{dx}$.
4. Solve for unknown and desired derivative.

Procedure. ***Logarithmic Differentiation***

Logarithmic differentiation is a technique used to find simplify our process of taking derivatives of complex algebraic function by relying on rules of logarithms:

1. Take natural log of both sides of equation $y = f(x)$
2. Use rules of logarithms to simplify expressions on both the left and right hand side of equality.
3. Differentiate implicitly both sides of equations with respect to x
4. Solve for unknown and desired derivative.