Lesson 9: Partial Derivatives Handout
Reference: Brigg's "Calculus: Early Transcendentals, Second Edition"
Topics: Section 12.4: Partial Derivatives, p. 894 - 907

Definition. *p. 897* ***Partial Derivative with respect to*** x

Let $f : D \to \mathbb{R}$ be a multivariable variable function, where $D = \mathbf{Dom}(f) \subseteq \mathbb{R}^2$. Then, the **partial derivative of a function** f **with respect to** x **at point** $(a, b) \in D$, is

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h} \qquad \text{(Limit definition in derivative notation)}$$

provided this limit exists.

Notation: The partial derivative at a point (a, b) can be denoted using any of the following notation:

$$f_x(a,b) = \frac{\partial f}{\partial x}(a,b) = \frac{\partial f}{\partial x}(a,b)\bigg|_{(a,b)} = \frac{\partial}{\partial x}\Big[\ f(x,y)\ \Big]\bigg|_{(a,b)}$$

The last two expressions rely on the "evaluated at" bar that we first used in integral calculus (Math 1B). Notice, to define this derivative, we apply the partial differentiation operator $\dfrac{\partial}{\partial x}$ with respect to x with

$$f_x(x,y) = \frac{\partial}{\partial x}\Big[\ f(x,y)\ \Big]$$

Definition. *p. 897* ***Partial Derivative with respect to*** y

Let $f : D \to \mathbb{R}$ be a multivariable variable function, where $D = \mathbf{Dom}(f) \subseteq \mathbb{R}^2$. Then, the **partial derivative of a function** f **with respect to** y **at point** $(a, b) \in D$, is

$$f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h} \qquad \text{(Limit definition in derivative notation)}$$

provided this limit exists.

Notation: The partial derivative at a point (a, b) can be denoted using any of the following notation:

$$f_y(a,b) = \frac{\partial f}{\partial y}(a,b) = \frac{\partial f}{\partial y}(a,b)\bigg|_{(a,b)} = \frac{\partial}{\partial y}\Big[\ f(x,y)\ \Big]\bigg|_{(a,b)}$$

The last two expressions rely on the "evaluated at" bar that we first used in integral calculus (Math 1B). Notice, to define this derivative, we apply the partial differentiation operator $\dfrac{\partial}{\partial y}$ with respect to y with

$$f_y(x,y) = \frac{\partial}{\partial y}\Big[\ f(x,y)\ \Big]$$

Definition. *p. 899* ***Higher-order partial derivative notation***

Table 12.4

Notation 1	Notation 1	What we say ...
$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$	$(f_x)_x = f_{xx}$	*d squared f dx squared* or *f-x-x*
$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$	$(f_y)_y = f_{yy}$	*d squared f dy squared* or *f-y-y*
$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$	$(f_y)_x = f_{yx}$	*f-y-x*
$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$	$(f_x)_y = f_{xy}$	*f-x-y*

Theorem 12.4. *p. 900* ***(Clairaut) Equality of Mixed Partial Derivatives***

Assume that $f(x, y)$ is a multivariable function with domain $D \subseteq \mathbb{R}^2$. Suppose that f_{xy} and f_{yx} are continuous throughout D. Then, for all points in D, we have

$$f_{xy}(x, y) = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\left[f(x, y)\right]\right] = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left[f(x, y)\right]\right] = f_{yx}(x, y)$$

Definition. *p. 902* ***Differentiability***

The function $z = f(x, y)$ is **differentiable at** (a, b) provided that $f_x(a, b)$ and $f_y(a, b)$ exist and the change $\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$ equals

$$\Delta z = f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \epsilon_1\,\Delta x + \epsilon_2\,\Delta y$$

where for a fixed a and b, the values of ϵ_1 and ϵ_2 depend only on Δx and Δy with $(\epsilon_1, \epsilon_2) \longrightarrow (0, 0)$ and $(\Delta x, \Delta y) \longrightarrow (0, 0)$

Theorem 12.5. *p. 903* ***Conditions for Differentiability***

Suppose the multivariable function $f(x, y)$ has partial derivatives f_x and f_y defined on an open set containing point (a, b) with f_x and f_y continuous at (a, b). Then, f is differentiable at (a, b).

Theorem 12.6. *p. 903* ***Differentiable Implies Continuous***

If the multivariable function $f(x, y)$is differentiable at (a, b), then, f is continuous at (a, b).