Linear Algebraic Nodal Analysis: An Applied Project for a First Course in Linear Algebra

Abstract: Many students who enroll in a first course in linear algebra major in STEM disciplines other than mathematics. Teachers who serve such students may find it difficult to provide authenic problems from these broader areas that ignite students' interest in linear algebra. In this paper, we highlight an interdisciplinary learning activity that engages students in using linear systems of equations to model the behavior of practical electric circuits. This exercise fits nicely into standard introductory linear algebra curricula and is designed to excite students majoring in engineering, physics, or applied mathematics. We also include references to a collection of open-access resources to support instructors who want to use this material in project-based, flipped-learning, inquiry-oriented, or independent-study environments.

Keywords: Mathematical modeling, matrix multiplication, nonsingular linear systems, block matrix, electric circuit analysis, nodal analysis

INTRODUCTION

In this paper, our goal is to empower students and faculty in introductory linear algebra courses to create rich learning experiences that inspire students to persist in STEM majors and build student excitement about their future academic and career endeavors. To do this, we explore an interdisciplinary project that includes a novel modeling approach to analyze electric circuits using matrix equations. We want to help students discover meaningful answers to the questions: "What are real-world applications of this theory?" and "How does this relate to my major?". Our hands-on modeling activity is matrix-oriented, incorporates technology, focuses on the needs and interests of students, and supports key client disciplines that require linear algebra as a prerequisite class. In other words, it addresses four of the five recommendations for a first course in linear algebra from the Linear Algebra Curriculum Study Group [9]. This applied project also provides an example to demonstrate that working with block matrix structures can make mathematical analysis easier by decoupling variables [23].

We have a national need to improve the quality of undergraduate STEM education "to ensure the economic strength, national security, global competitiveness, environment, and health of the United States" [35, p. 7]. As is outlined in the 2012 Report for the President's Council of Advisors on Science and Technology (PCAST), "fewer than 40 percent of students who enter college intending to major in a STEM field complete college with a STEM degree" [39, p. i]. The PCAST report argues that we must increase the number of STEM majors we recruit and retain by focusing on improving STEM courses taken in the first two years of college while placing a special priority on utilizing empirically validated teaching strategies by emphasizing active learning and student engagement.

Too many introductory STEM classes are taught using lectures in which the instructor delivers long, technical monologues and provides little opportunity for student engagement. Extensive research indicates that to increase the number of STEM graduates, college teachers should be "abandoning traditional lecturing in favor of active learning" [20]. Indeed, 90% of students cite "poor teaching" and "problems with instructor pedagogy" as powerful concerns that contribute to their decision to leave STEM fields [44, p. 8 - 9]. These students indicate that they find "it hard to retain their interest in the subject where instructors failed to present the material in a stimulating manner" [44, p. 10]. Students also report that they yearn for and often do not find illustrations of how course content can be applied to authentic problems related to their academic and career interests [44, p. 10].

College teachers may struggle to find and use curriculum that is designed to tap into their students' interests due to the design of traditional textbooks. Many introductory mathematics textbooks in the United States "base their whole approach on the idea of isolating methods, reducing them to their simplest form, and practicing them" which "induces boredom" and gives "the most simplified and disconnected version of the method to be practiced" so that students have "no sense of when or how they might use the method" [8, p. 42]. Textbooks that do provide applications often focus on simplified models of ideal, theoretic situations that are far removed from students' lived experiences. Students that engage with these types of learning materials likely have no opportunity to collect realistic data, no chance to discover connections with the material world, and may find it impossible to discover how the math they learn in class is applicable to solving problems they care about.

In recognition of the urgent need for drastic changes in mathematics curricula and teaching practices, a wide range of mathematics organizations have produced studies, reports, and guidelines offering specific recommendations for improving undergraduate mathematical sciences programs. The Mathematical Association of America (MAA) argues that "the status quo is unacceptable" and urges teachers to update introductory mathematics curricula to focus on active learning, to use evidence-based teaching practices, and to establish stronger interdisciplinary learning tasks that ignite students' interests [30]. The MAA also suggests that "students should learn to link applications and theory" and "develop mathematical independence and experience open-ended inquiry" [11].

The Society for Industrial and Applied Mathematics (SIAM) along with the Consortium for Mathematics and Its Applications (COMAP) provide convincing arguments for incorporating real-world mathematical modeling activities into classroom learning in their Guidelines for Assessment and Instruction in Mathematical Modeling Education (GAIMME) report [22]. These guidelines emphasize that mathematical modeling "should be taught at every stage of a student's mathematical education" and should "be used to motivate curricular requirements" while highlighting "the importance and relevance of mathematics in answering" questions that students care about. In 2013, the National Research Council (NRC) recommended that introductory college mathematics courses should provide students with opportunities to make explicit connections between mathematics and other disciplines to better "understand the role of the mathematical sciences in the wider world of science, engineering, medicine, defense, and business" [36, p. 2 - 3]. Similar guidance is offered by the Transforming Post-Secondary Education in Mathematics (TPSE Math) project which calls for increased inclusion of modeling in college math classes that "demonstrate ways to connect the mathematics studied to students intended majors" while also offering "interesting ways to deliver instruction and engage students" [51, p. 5]. These national reports and professional organizations declare that we must re-imagine our mathematics curriculum to include interesting mathematical modeling activities that invite students to work on real-world problems. They call on us to empower students to discover for themselves that our course content is useful, meaningful, and powerful, particularly in introductory mathematics classes that set a foundation for more advanced learning in many STEM fields.

The modeling activity and active learning exercises (see Appendix A) presented here, along with other related projects [2], are designed to meet this exact need. However, developing curriculum is half of the puzzle. The other half involves figuring out how to implement this type of activity inside the classroom using empirically-validated, research-based teaching strategies. To answer questions about how to design teaching policies that center active learning using applied modeling activities, the research cited below indicates that we can increase student persistence by implementing three interventions in introductory STEM courses.

The first intervention relates to the design of teaching routines that intellectually engage students in meaningful learning activities. Replacing traditional lectures with active learning tasks, like the ones described in this paper, has been shown to improve student learning, increases retention rates, and reduce achievement gaps within diverse student populations [31]. The key to active learning is to engage students in "doing things and thinking about what they are doing" [7, p. 2]. Such active learning tasks encourage students to engage in critical thinking, creative problem solving, attentive observation, meta-cognitive reflection, meaning making, and relationship building in the context of shared learning [18, p. 115 – 137].

The second intervention involves creating relationship-rich educational experiences for students. Decades of research suggests that student-faculty, peer-to-peer, and student-staff relationships are essential factors that lead to deeper learning, a stronger sense of belonging, and higher achievement in college. Indeed, "students' interactions with peers, faculty, and staff strongly influence the breadth and depth of student learning, retention and graduation rates, and a wide range of other outcomes, including critical thinking, identity development, communication skills, and leadership abilities" [17, p. 5]. Students who enjoy authentic relationships and identify with a community of STEM professionals persist longer and show reduced departure rates from STEM fields [15].

The third intervention that affects student persistence involves an explicit focus on tapping into students' intrinsic motivations for learning [40]. A powerful source of intrinsic motivation is *interest* which consists of two distinct sentiments including "an individual's momentary experience of being captivated by an object as well as more lasting feelings that the object is enjoyable and worth further exploration" [24]. Research shows that when students feel interested in learning, they spend more time studying, learn at a deeper level, persist longer on learning tasks, and get better grades in their classes [46]. Interest also plays a central role in the growth of expertise and knowledge development [43].

DELIVERY

The applied project described in this paper can be successfully integrated into introductory linear algebra courses to address the three interventions described above. The first author of this work uses this modeling activity in flipped-learning environments [50] where the teacher and students work together to develop the expectation that students are responsible to get exposure to new material outside of class. Students either watch YouTube videos [3] that cover course content or read an instructor-authored textbook manuscript that aligns with the videos and are typeset in LaTeX. These open-access resources are designed to help students build the skills they need to complete their chosen learning exercises. In this setup, each student is tasked to create a comprehensive learning portfolio [53] that includes not only detailed notes to document their understanding of course content but also solutions to any exercises that students find interesting. At their best, these exercises are open-ended, extend beyond the content covered in the videos, are structured to scaffold learning, allow for multiple solution paths [8, pp. 57 – 91], and guide students to organize their thinking and deepen their explorations [1, pp. 40 – 65].

This flipped-learning structure ensures that in-class meetings are dedicated to active learning in small groups. While students engage in individual and group problem solving during class, the teacher moves around the room to build relationships with students, provide support, give feedback, and answer questions. As students progress in the course, they level-up to the entire modeling experience described in this paper. For the capstone project of the academic term, students use the algorithm described below to analyze an example electric circuit and also provide answers to questions from Appendix A that they find interesting [10]. Students include their work on this applied modeling project in their learning portfolio.

This approach may require support in topics that are not traditionally included in mathematics curriculum. To help the reader, we provide a collection of explanatory resources on the companion website for this paper [4]. These include introductory workbooks on how to build and measure electric circuits, resources for model verification using the free online MultiSim software [34], YouTube videos to support on-demand direct instruction, example circuit problems, MATLAB script files, and more. These materials can be used to support fun independent study opportunities, to guide modeling activities in project-based or flipped learning classrooms, or to build an entire course around the archetype problem of circuit analysis.

The rest of this paper is organized to support the reader in mastering the mathematical foundations of the linear algebraic nodal analysis (LANA) algorithm. An elementary familiarity with the application domain is all that's needed to use this activity successfully. However, for readers who seek deep understanding, we briefly review the tradition of using linear algebra to model electric circuits. We then describe each step of the LANA algorithm in general and explore the output of each step for the example circuit given in Figure 1 below. Appendix A includes open-ended mathematical inquiry tasks [8, p. 57 – 91] that teachers and students can use to inspire independent explorations and collaborative sense-making activities [33].

ALGORITHM STATEMENT

Many popular undergraduate- and graduate-level textbooks in applied linear algebra use electric circuit theory as a motivating example [6, 12, 16, 19, 29, 32, 48]. Some authors have suggested an equilibrium equation framework in the form

$$A^T C A\mathbf{x} = A^T C\mathbf{b} - \mathbf{f} \tag{1}$$

to describe the behavior of electrical networks [37, 49]. However, these authors do not provide a general approach to transform practical problems in circuit analysis into their suggested equilibrium equation framework. To address this issue, we propose the linear-algebraic nodal analysis (LANA) algorithm. This algorithm takes as input any linear resistor network that includes resistors, ideal voltage sources, and ideal current sources and produces as output a nonsingular linear-systems problem in the desired equilibrium equation framework (1). Below is the seven-step LANA algorithm to transform the problem of analyzing the electric behavior of any linear resistor circuit into a system of equations involving a nonsingular matrix in the exact equilibrium equation framework (1) that we desire.

Algorithm 1 The linear-algebraic nodal analysis (LANA) algorithm

1: Identify and label all circuit nodes.
2: Model the circuit as a directed graph.
3: Create all circuit matrices.
4: State the entire set of circuit equations.
   4.1: State the equations from Kirchhoff's current laws (KCLs).
   4.2: State the equations from the branch constitutive relations (BCRs).
   4.3: State the equations from Kirchhoff's voltage laws (KVLs).
   4.4: Combine the circuit equations.
5: Identify ordinary and generalized nodes.
6: Create a minimal set of independent node potentials.
   6.1: Impose one constraint for each voltage source.
   6.2: Impose one constraint for the ground node.
7: Solve the equilibrium equation for the circuit.

Our approach in the LANA algorithm makes explicit connections between standard introductory linear algebra content, curriculum from introductory physics courses on electricity and magnetism, and the classical nodal analysis algorithm [13, pp. 108-126] that is popular in almost all introductory circuit analysis courses. This also sets a foundation for the modified nodal analysis algorithm [25] which is implemented by many circuit simulation software packages [41, 52]. In our work to develop the LANA algorithm, the first author created a novel linear-algebraic-based proof that the coefficient matrix arising in classical nodal analysis is nonsingular (please contact that author for more details). Finally, following trends in the numerical simulation of electric networks [21], engineers, scientists, and applied mathematicians can use our procedure to describe problems in circuit analysis via block matrices. To help readers understand the notation used in the LANA algorithm and organize the various equations that arise, the support website for this paper includes a guide for all variables that show up in this work [4].

An example of this algorithm applied to the circuit in Figure 1 is included below. The LANA algorithm applies to circuits, like those in Figure 1, containing only *two-terminal elements* each of which touches two distinct circuit nodes. To begin, we need a complete description of the type, magnitude, and interconnectivity of each electrical component included in our circuit. In visual form, this information can be encoded in an ideal circuit schematic as is given in Figure 1.

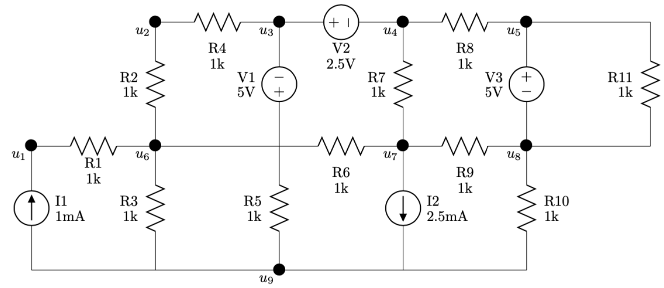

Figure 1: An ideal schematic diagram of a circuit containing eleven resistors, three ideal voltage sources, each of which provides a constant voltage drop across the two leads, and two ideal current sources, each of which provides a constant current through the two leads. We use this example to illustrate how this LANA algorithm works in practice.

1 Step 1: Identify and label all circuit nodes

The first step in our algorithm introduces a strategy to identify and label the nodes in our circuit. To do so, we take advantage of the *node identification heuristic* in which we manipulate the original circuit by removing the bodies of each ideal circuit element. The result is a skeleton consisting only of contiguous segments of conductive material. Figure 2 captures the output of this heuristic applied to the example circuit in Figure 1.

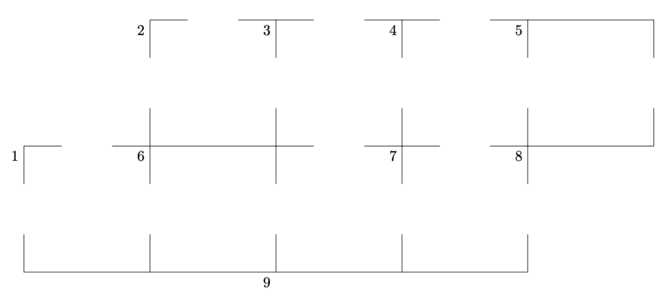

Figure 2: The skeleton circuit of contiguous wire segments that results from the node identification heuristic applied to the circuit from Figure 1. Notice the body of each circuit element is replaced with a blank space.

Each contiguous segment of wire in Figure 2 is represented by a unique node of the circuit. Our numbering scheme roughly moves from left to right and top to bottom using consecutive natural numbers. As shown in Figure 2, our example circuit contains nine unique circuit nodes.

2 Step 2: Model the circuit as a directed graph

The second step of the LANA algorithm is to model the circuit as a directed graph. This *digraph* model $\mathcal{G} = (\mathcal{N},\ \mathcal{E})$ consists of two sets $\mathcal{N}$ and $\mathcal{E}$. By translating our ideal schematic into a digraph, we form a bridge between the original circuit design and the linear-algebraic description of that circuit.

2.1 The Set $\mathcal{N}$: The nodes of digraph $\mathcal{G}$

The set $\mathcal{N}$ contains a finite list of *nodes* in the digraph $\mathcal{G}$. Each digraph node $k \in \mathcal{N}$ corresponds to a unique circuit node identified in step 1. We label the digraph nodes using ascending consecutive natural numbers with $\mathcal{N} = \{1,\ 2,\ ...,\ n_g\}$ where $n_g \in \mathbb{N}$ represents the total number of nodes in the circuit. In our example circuit, we see $n_g = 9$.

2.2 The Set $\mathcal{E}$: The edges of digraph $\mathcal{G}$

The set $\mathcal{E} \subset \mathcal{N} \times \mathcal{N}$ contains a finite list of *directed edges* where each edge $e \in \mathcal{E}$ in our digraph represents a single ideal circuit element. Borrowing from graph theory [14], each directed edge e is written as an ordered pair of nodes in the form $e = (\text{initial node}, \text{terminal node})$. In the context of simple electric circuits, no directed edge extends from a node to itself, so we never see the self loop (k, k) as a directed edge in this context.

We enumerate the directed edges using ascending consecutive natural numbers. To count the total number of edges contained in our digraph, we say m_r represents the number of resistors in our circuit, m_v enumerates the number of ideal voltage sources, and m_i counts the number of ideal current sources in our circuit. An ideal voltage source (think of this like a battery) provides energy to the circuit via a constant voltage drop across the two leads over a range of output currents. On the other hand, an ideal current source provides energy to the circuit using a constant current flow through the two leads over some range of output voltages.

Electrical engineers use ideal voltage and current sources as part of a wide collection of circuit analysis techniques and theorems that, when applied skillfully, can accurately predict the electric behavior of real circuits used in engineering applications. Ideal voltage and current sources are mathematical models that do not exist in the physical world [45, p. 62 - 65]. However, there are numerous practical circuits that approximate the behavior these ideal circuits for specific intended use cases. It is worth noting that the first author of this paper teamed up with an engineering team to build physical circuit devices that transform a 9-volt battery into an ideal voltage or ideal current source for in-class use so that students can prototype and measure physical circuits for comparison against the modeled outputs that they produce using linear algebraic theory [5].

As a practical matter, a circuit that effects an action does so through resistance to current, so m_r will always be positive, but m_v and m_i can be 0. The total number of edges in our digraph model equals the total number of circuit components and is given by $m = (m_r + m_v + m_i)$. Moreover, we choose a special enumeration scheme to label each edge. We begin by counting and labeling edges $e_1, e_2, ..., e_{m_r}$ corresponding to the resistors in our circuit. Next, assuming $m_v \geq 1$, we continue our count by labeling the edges corresponding to ideal voltage sources as edges $e_{m_r+1}, ..., e_{m_r+m_v}$. If $m_i \geq 1$, we conclude by labeling the edges that represent current sources as edges $e_{m_r+m_v+1}, ..., e_m$. The only condition required in each count is that we have a bijection between circuit elements and graph edges. Although no specific subordering is necessary, we recommend counting from left to right and top to bottom when possible. By enumerating our edges in this way, we draw a useful partition of the matrices and vectors that model our circuit.

Since we're creating a directed graph model for our circuit, we assign a direction to each edge. To do so, we follow three conventions.

1. An edge that models an ideal voltage source points in the direction from the positive + lead toward the negative − lead of that source.
2. An edge that models an ideal current source points in the same direction as that current source.
3. An edge corresponding to a resistor can be assigned any direction since the voltage across and current through each resistor are not predefined.

Readers familiar with engineering conventions might impose the optional constraint that each node should have at least one edge that points inward and one edge that points outward, though this is unnecessary in our general framework.

2.3 Developing the digraph

Applying this framework to the example circuit in Figure 1, we see that $m_r = 11$, $m_v = 3$, $m_i = 2$, and $m = 16$. This leads to an edge enumeration scheme in which edges 1 through 11 correspond to the 11 resistors and edges 12 through 14 correspond to the three ideal voltage sources. The remaining edges e_{15} and e_{16} correspond to the two ideal current sources. We follow the three conventions listed above to choose a direction for each edge. This yields a digraph model in Figure 3 below for the circuit provided in Figure 1.

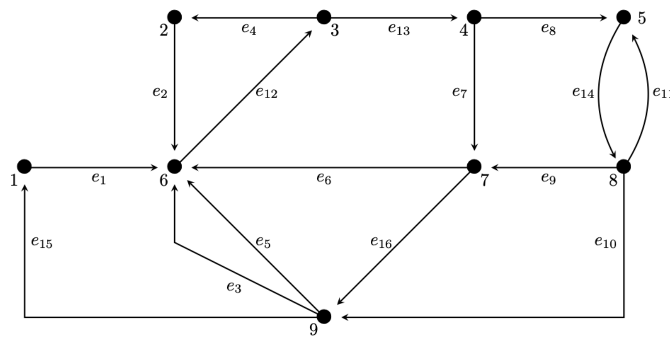

Figure 3: A digraph model for the example circuit in Figure 1 with ideal circuit elements replaced by directed edges using the enumeration and direction conventions described above.

In Figure 3, we see that directed edge $e_1 = (1, 6)$ models resistor R1. This edge initializes at node 1 and terminates at node 6.

3 Step 3: Create all circuit matrices

The third step of the LANA algorithm is to create all circuit matrices including the *incidence matrix* $A_g \in \mathbb{R}^{m \times n_g}$, the *node voltage potential vector* $\mathbf{u}_g \in \mathbb{R}^{n_g}$, the *voltage-drop vector* $\mathbf{v} \in \mathbb{R}^m$ and the *current vector* $\mathbf{i} \in \mathbb{R}^m$.

3.1 The incidence matrix

Using the digraph model of the circuit from step 2, we bridge the gap between the visual schematic and the linear-algebraic description of the circuit. Recall that we consider circuits in which each ideal element is connected to two distinct nodes. We encode the connectivity between the nodes via the directed edges in the *incidence matrix* $A_g \in \mathbb{R}^{m \times n_g}$. Our entry-by-entry definition of this matrix is given as

$$a_{jk} = \begin{cases} 1 & \text{if edge } e_j \text{ leaves node } k, \\ -1 & \text{if edge } e_j \text{ enters node } k, \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

for $j = 1, 2, ..., m$ and $k = 1, 2, ..., n_g$. Notice that in the incidence matrix A_g, the rows correspond to the edges in our digraph while the columns map to the graph's nodes. In our entry-by-entry definition (2), each row of A_g has two nonzero entries and summing all columns together produces the zero vector.

Our definition of incidence matrix A_g departs from a popular convention for defining incidence matrices for directed graphs. Many authors define incidence matrices for directed graphs where the nodes of the digraph correspond to rows of the matrix and the edges correspond to columns. That common convention produces the transpose of the matrix we use in this paper. Given that the goal of the LANA algorithm is to output a system in the equilibrium equation framework (1), we define our incidence matrix A_g to encode each edge as a row and track each node via the column numbers of A_g. This approach produces matrices in the exact form that we desire with the transpose operator in the proper place.

The edge enumeration scheme from step 2 yields a convenient block partition of this incidence matrix in the form:

$$A_g = \begin{bmatrix} A_{r_g} \\ \hdashline A_{v_g} \\ \hdashline A_{i_g} \end{bmatrix} \tag{3}$$

where $A_{r_g} \in \mathbb{R}^{m_r \times n_g}$, $A_{v_g} \in \mathbb{R}^{m_v \times n_g}$, and $A_{i_g} \in \mathbb{R}^{m_i \times n_g}$. Recall that the subscripts *r, v,* and *i* denote rows of our incidence matrix A_g corresponding to groups of resistors, voltage sources, and current sources, respectively. We use this partition later to analyze each block independently. Applying the entry-by-entry definition (2) to our example graph in Figure 2, we produce the incidence matrix

$$A_g = \begin{bmatrix} \\ \\ \\ \\ A_{r_g} \\ \\ \\ \\ \hdashline A_{v_g} \\ \hdashline A_{i_g} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ \hdashline 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ \hdashline -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix}.$$

To finish step 3, we create vectors to describe the circuit's electronic behavior.

3.2 The node voltage potential vector

Each node in the circuit has a corresponding voltage potential. Because the current running through each resistor in the circuit is governed by changes in voltage across each resistor (as described by Ohm's law), we write the node voltages relative to a designated *ground node* or *datum node*. We can select any of the n_g nodes as ground, which amounts to choosing the location where we attach the negative (black) lead of a multimeter to measure voltage potentials [47]. To capture this information in vector form, we create the *node voltage potential vector* $\mathbf{u}_g \in \mathbb{R}^{n_g}$, where entry u_k measures the ideal voltage potential at node $k \in \mathcal{N}$. Notice that both matrices A_g

and $\mathbf{u}_g$ include a subscript g to indicate that these structures reference the entire set of nodes, including the user-selected ground node.

3.3 The voltage-drop vector and current vector

From engineering fundamentals, each circuit element can be described by the voltage-drop across and the current flowing through that element. We capture these characteristics in the voltage and current vectors given by

$$\mathbf{v} = \begin{bmatrix} \mathbf{v}_r \\ \hdashline \mathbf{v}_v \\ \hdashline \mathbf{v}_i \end{bmatrix} \quad \text{and} \quad \mathbf{i} = \begin{bmatrix} \mathbf{i}_r \\ \hdashline \mathbf{i}_v \\ \hdashline \mathbf{i}_i \end{bmatrix}, \tag{4}$$

where $\mathbf{v}_r, \mathbf{i}_r \in \mathbb{R}^{m_r}$, $\mathbf{v}_v, \mathbf{i}_v \in \mathbb{R}^{m_v}$, and $\mathbf{v}_i, \mathbf{i}_i \in \mathbb{R}^{m_i}$ so that $\mathbf{v}$ and $\mathbf{i}$ have the same number of rows as the incidence matrix A_g. These block partitions come from the edge enumeration scheme in step 2 and match the row dimensions for the blocks that define the incidence matrix A_g.

4 Step 4: State the entire set of circuit equations

Let's count the number of unknown variables we have in our system. The vectors $\mathbf{i}_r$ and $\mathbf{v}_r$ both include m_r entries so that, combined, these vectors account for $2\,m_r$ variables. The vector $\mathbf{v}_i$ has m_i entries while the vector $\mathbf{i}_v$ has m_v entries. Finally, the vector $\mathbf{u}_g$ has n_g variables. Thus, at this point in our algorithm, our systems has a total of $\ell = 2 \; m_r + m_i + m_v + n_g$ unknown variables. We note that the coefficients in the $\mathbf{v}_v$ and $\mathbf{i}_i$ blocks are the known values of the ideal voltage and ideal current sources, respectively. Thus, the $m_v + m_i$ variables corresponding to these two blocks are pre-defined by the structure of the circuit while all other quantities in the vectors $\mathbf{v}$ and $\mathbf{i}$ are unknown. In steps $4 - 6$ of the LANA algorithm, we simplify our analysis problem by imposing constraints amongst the ℓ unknowns to produce a minimal set of n variables from which all other values can be calculated. For the circuit in Figure 1, this yields a reduction from $\ell = 36$ to $n = 5$ variables.

The fourth step of the LANA algorithm is to state the entire set of circuit equations which come from Kirchhoff's current laws (KCLs), branch constitutive relations (BCRs), and Kirchhoff's voltage laws (KVLs). By combining these equations together, we achieve the first of two different reductions processes that eventually result in our minimal set of unknown circuit variables.

4.1 State the equations from Kirchhoff's current laws

The first set of circuit equations come from *Kirchhoff's current laws* (KCLs) which states that the sum of all currents running into and out of any node must be zero.

Since there is one KCL equation for each node in a circuit, we have a total of n_g unique KCL equations that can be written in matrix form as

$$A_g^T\,\mathbf{i} = \mathbf{0}, \tag{5}$$

where the right-hand side signifies a zero column vector with n_g rows. Using the block version (3) of the incidence matrix A_g and current vector $\mathbf{i}$ (4), we rewrite these KCLs in *block form* as

$$A_{r_g}^T\,\mathbf{i}_r + A_{v_g}^T\,\mathbf{i}_v + A_{i_g}^T\,\mathbf{i}_i = \mathbf{0}. \tag{6}$$

Since $\mathbf{i}_i$ is a known quantity provided by our DC current sources, we bring this term on the right-hand side yielding

$$A_{r_g}^T\,\mathbf{i}_r + A_{v_g}^T\,\mathbf{i}_v = -A_{i_g}^T\,\mathbf{i}_i, \tag{7}$$

where $\mathbf{i}_r$ and $\mathbf{i}_v$ are two unknown vector that we deal with in the next steps below.

4.2 State the equations from the branch constitutive relations

To eliminate explicit reference to the $\mathbf{i}_r$ vector, we use *branch constitutive relations* (BCRs). For *resistive networks* that contain only resistors along with ideal voltage and ideal current sources, the BCRs come in the form of *Ohm's law* which describes a linear relationship between the current flowing through and the voltage drop across a resistor. We capture this information for all resistors in the circuit using a matrix equation via either the *resistance* or *conductance* forms of Ohm's law given by

$$\mathbf{v}_r = R\,\mathbf{i}_r \qquad \text{or} \qquad \mathbf{i}_r = G\,\mathbf{v}_r. \tag{8}$$

The *resistance matrix* $R \in \mathbb{R}^{m_r \times m_r}$ is a diagonal matrix whose jth diagonal entry is equal to r_j which is the resistance value of the jth resistor, where $j = 1, 2, ..., m_r$. The *conductance matrix* $G = R^{-1}$. To state our equations with a minimal number of unknown variables, we replace $\mathbf{i}_r$ in the block KCL equations (7) with the equation (8) for the conductance form of Ohm's law involving matrix G. This yields the partially reduced equation

$$A_{r_g}^T G\,\mathbf{v}_r + A_{v_g}^T\,\mathbf{i}_v = -A_{i_g}^T\,\mathbf{i}_i. \tag{9}$$

From here, we further simplify our work by identifying the relationship between the voltage drop vector $\mathbf{v}_r$ and node potential vector $\mathbf{u}_g$.

4.3 State the equations from Kirchhoff's voltage laws

To connect the voltage drop and node potential variables, we use *Kirchhoff's voltage laws* (KVLs) in node potential form. These equations state that the voltage drop across each two-terminal element is calculated by the difference between the node voltage potentials at each terminal. Because the topology of our circuit is encoded in the incidence matrix A_g, we write the KVLs in matrix form as

$$A_g \, \mathbf{u}_g = \mathbf{v}. \tag{10}$$

Just as we write the KCLs in block form (6), we use the step 2 enumeration scheme and block versions of our circuit matrices to write *element-specific KVLs* given by

$$A_{r_g} \, \mathbf{u}_g = \mathbf{v}_r, \qquad A_{v_g} \, \mathbf{u}_g = \mathbf{v}_v, \qquad \text{and} \qquad A_{i_g} \, \mathbf{u}_g = \mathbf{v}_i. \tag{11}$$

These KVLs show that as soon as we find the vector $\mathbf{u}_g$, matrix-vector multiplication produces the values of $\mathbf{v}_r$ and $\mathbf{v}_i$.

4.4 Combine the circuit equations

Using the resistor KVLs (11), we replace the vector $\mathbf{v}_r$ in the partially reduced equation (9) with $A_{r_g}\mathbf{u}_g$ yielding

$$A_{r_g}^T G A_{r_g} \mathbf{u}_g + A_{v_g}^T \mathbf{i}_v = -A_{i_g}^T \mathbf{i}_i. \tag{12}$$

In this reduced system, we have two categories of data as shown in Table 1.

Known quantities	Unknown quantities
A_{r_g}, A_{v_g}, A_{i_g}, G, $\mathbf{v}_v$, $\mathbf{i}_i$	$\mathbf{u}_g$, $\mathbf{i}_v$

Table 1: Known and unknown quantities

By combing the KCLs, BCRs, and KVLs together, we decrease the initial ℓ unknown variables to a smaller set of $(n_g + m_v)$ unknowns. For the circuit in Figure 1, this reduces $\ell = 36$ to $(n_g + m_v) = 12$ variables. We further reduce the number of variables needed to analyze the circuit by identifying constraints within the remaining unknown variables.

5 Step 5: Identify ordinary and generalized nodes

Each voltage source in a circuit creates a constraint between the node potential variables on either side of that source. We borrow engineering notation to track which nodes are connected to voltage sources and which are not [13, pp. 118 – 119]. An *ordinary node* is a single node that does not touch a voltage source while a *generalized node* is a set of circuit nodes that are linked together by voltage sources.

To identify ordinary and generalized nodes, we use the *deactivated circuit heuristic* in which we deactivate the power sources. To deactivate a voltage source, we set the value of that source to zero which implies there is no voltage potential change across the leads of that voltage source (also known as a short circuit). To deactivate a current source, we set the value of that source to zero which implies zero current can flow between the leads of that element (also known as an open circuit). From this perspective, deactivating the power sources is equivalent to drawing a *deactivated resistor network* where we replace the voltage sources with short circuits and current sources with open circuits. Figure 4 presents the deactivated resistor network for the circuit from Figure 1.

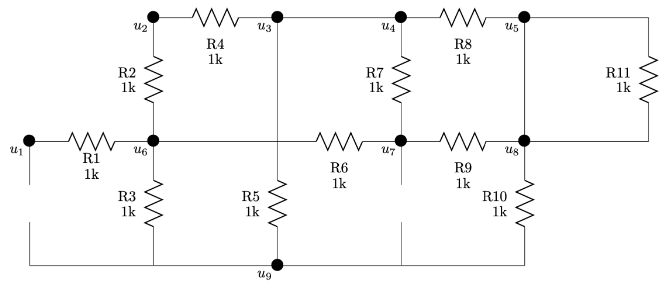

Figure 4: The deactivated resistor network for the example circuit from Figure 1. Deactivation involves two steps: (1) Replace each voltage source with a short circuit which we draw as a line segment by replacing the body of each voltage source with a straight line between the corresponding nodes. (2) Replace each current source with an open circuit which is drawn as a gap between the corresponding nodes by erasing the body of each current source.

When we deactivate the circuit, a set of nodes that are linked via voltage sources meld together to form a single generalized node. Table 2 below presents these node classifications for the circuit in Figure 1.

Node Classification	Set of node indices	Node variables
Ordinary nodes	$\{1\}, \{2\}, \{7\}, \{9\}$	u_1, u_2, u_7, u_9
Generalized node 1	$\{3, 4, 6\}$	u_3, u_4, u_6
Generalized node 2	$\{5, 8\}$	u_5, u_8

Table 2: Classification of ordinary and generalized nodes

The deactivated circuit heuristic helps to elucidate key connectivity features within the circuit. The process of deactivating each voltage source merges two nodes together thus eliminating m_v nodes from the original circuit. In contrast, deactivating the current sources does not change the number of nodes. What is left after the deactivation process is a circuit containing only resistors that are linked together via a reduced set of $(n_g - m_v)$ nodes.

We say that the deactivated resistor network is *connected* if we can find a path of resistors between any two nodes in that circuit. We note that in a connected network, removing edges correspond to current sources does not disconnect the digraph model of the circuit, as can be seen in Figure 3 for the circuit in this article. The deactivated network in Figure 4 satisfies this definition. In general, as long as we don't have two subcircuits that are attached together by one or more current sources, the deactivated resistor network will always be connected. This connectivity feature guarantees a unique solution to the matrix equation (1) generated by the LANA algorithm (contact the first author for a novel proof of existence and uniqueness).

6 Step 6: Create minimal set of independent node potentials

The LANA algorithm uses linear algebra to reduce the vector $\mathbf{u}_g$ to a minimal set of independent node variables from which all other quantities in the circuit can be calculated. To achieve this reduction, we partition the entries of $\mathbf{u}_g$ into two lists. The first list of *constrained variables* includes one node potential variable for each of the m_v voltage sources and exactly one additional variable for our chosen ground node. The second list of *independent variables* are the chosen node voltage potentials that remain after eliminating the constrained quantities.

To impose the $(m_v + 1)$ constraints, we look at two features of our modeling problem. The first set of constraints is encoded in the element-specific KVLs (11). The voltage-source KVLs form a linear-systems problem

$$A_{v_g}\,\mathbf{u}_g = \mathbf{v}_v \tag{13}$$

since the vector $\mathbf{v}_v$ on the right-hand side has known entries. This *voltage source general linear-systems problem* (13) encodes m_v restrictions amongst the entries of $\mathbf{u}_g$. The second type of constraint relates to our choice of ground node. To ground our circuit, we pick a reference node and set one entry of the vector $\mathbf{u}_g$ to zero. By eliminating all constraints among the entries of $\mathbf{u}_g$, we create a minimal list $\mathbf{u} \in \mathbb{R}^n$ of independent node variables, where $n = (n_g - m_v - 1)$.

To reduce $\mathbf{u}_g$ down to $\mathbf{u}$, we can impose the $(m_v + 1)$ constraints in either order. For example, we can first impose the m_v voltage-source constraints and then choose a ground node. Alternatively, we can first choose a ground node and then impose the m_v voltage source constraints. In this exposition, we choose to first impose the voltage-source constraints and then choose a ground node from the remaining variables. Either approach produces the same coefficient matrix that arises from classical nodal analysis [13, pp. 139 – 143]. Let's explore this reduction in practice.

6.1 Impose one constraint for each voltage source

The voltage source KVLs (13) come in both matrix and scalar forms. For the example circuit in Figure 1, we see:

$$A_{v_g}\,\mathbf{u}_g = \mathbf{v}_v \qquad\Leftrightarrow\qquad \begin{bmatrix} u_6 - u_3 \\ u_3 - u_4 \\ u_5 - u_8 \end{bmatrix} = \begin{bmatrix} v_{v_1} \\ v_{v_2} \\ v_{v_3} \end{bmatrix} = \begin{bmatrix} 5.0 \\ 2.5 \\ 5.0 \end{bmatrix} \tag{14}$$

Starting here, we produce a complete solution with $(n_g - m_v)$ *free variables* and m_v constraints. No matter what choice we make for the free variables, we can always produce a complete solution to the voltage-source KVLs (13) in the form

$$\mathbf{u}_g = \mathbf{p}_g + Z_{v_g}\mathbf{u}_f \tag{15}$$

where $\mathbf{p}_g \in \mathbb{R}^{n_g}$ is a particular solution, $\mathbf{u_f} \in \mathbb{R}^{n_f}$ is the vector of free variables [32, pp. 64 – 70], and the dimension $n_f = n_g - m_v$ represents the *number of free variables* from the voltage-source KVL equation (13). The columns of $Z_{v_g} \in \mathbb{R}^{n_g \times n_f}$ form a basis for null(A_{v_g}) (for details, contact the first author). Below are two different ways to do this for our example circuit from Figure 1.

(i) Let's start with the first generalized node. If we choose u_3 as the free variable, then $u_6 = u_3 + v_{v_1}$ and $u_4 = u_3 - v_{v_2}$. For the second generalized node, if u_5 is the chosen free variable, then $u_8 = u_5 - v_{v_3}$. The resulting complete solution to the voltage-source KVLs (13) is

$$\underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{bmatrix}}_{\mathbf{u}_g} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_3 - v_{v_2} \\ u_5 \\ u_3 + v_{v_1} \\ u_7 \\ u_5 - v_{v_3} \\ u_9 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ -v_{v_2} \\ 0 \\ v_{v_1} \\ 0 \\ -v_{v_3} \\ 0 \end{bmatrix}}_{\mathbf{p}_g} + \underbrace{\begin{bmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&0&1 \end{bmatrix}}_{Z_{v_g}} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_5 \\ u_7 \\ u_9 \end{bmatrix}}_{\mathbf{u}_f} \tag{16}$$

(ii) If we choose u_4 as the free variable for generalized node 1, then $u_3 = u_4 + v_{v_2}$ and $u_6 = u_4 + v_{v_2} + v_{v_1}$. If u_8 is the free variable for generalized node 2, then $u_5 = u_8 + v_{v_3}$ and the complete solution is

$$\underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{bmatrix}}_{\mathbf{u}_g} = \begin{bmatrix} u_1 \\ u_2 \\ u_4 + v_{v_2} \\ u_4 \\ u_8 + v_{v_3} \\ u_4 + v_{v_1} + v_{v_2} \\ u_7 \\ u_8 \\ u_9 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ v_{v_2} \\ 0 \\ v_{v_3} \\ v_{v_1} + v_{v_2} \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{p}_g} + \underbrace{\begin{bmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&0&1&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{bmatrix}}_{Z_{v_g}} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_4 \\ u_7 \\ v_8 \\ u_9 \end{bmatrix}}_{\mathbf{u}_f}$$

6.2 Impose a single constraint for the ground node

To impose the final constraint and produce our minimal list of independent node potentials, we choose a single ground node from the remaining $(n_g - m_v)$ free variables. Let's select option (i) in Section 6.1 with chosen free variables u_3 and u_5. We then designate any of the remaining nodes as ground. The moment we choose the ground node, the corresponding entry of the vector $\mathbf{u}_f$ is set to zero and shifts from unknown to known. Since the value of the ground node variable is set to zero, reference to this variable in the complete solution to the voltage-source KVLs (15) represents multiplication by zero. But, multiplying a column of a matrix by zero is equivalent to deleting the column from the resulting linear combination. In other words, choosing a ground node and setting the corresponding entry of $\mathbf{u}_f$ to zero is algebraically equivalent to a dimension reduction realized using multiplication with a matrix $D_{f_0} \in \mathbb{R}^{n_f \times n}$. The positive integer

$$n = n_f - 1 = (n_g - m_v) - 1$$

represents the minimum number of node voltage potentials needed to completely analyze the circuit.

We refer to this process of strategically deleting columns (or rows) of a matrix as *deflation*. Similarly, we say that a *deflation matrix* is any matrix used to delete or *deflate* the columns (or rows) of another matrix. We use the deflation matrix D_{f_0} to define

$$\mathbf{u} = D_{f_0}^T \, \mathbf{u}_f \qquad \text{and} \qquad Z = Z_{v_g} \, D_{f_0}. \tag{17}$$

The matrix D_{f_0} is formed by taking the $n_f \times n_f$ identity matrix and deleting the column corresponding to the chosen ground node as seen below.

A. Assume we ground node 9 and set $u_9 = 0$. We realize this constraint using the matrix equation

$$\underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_5 \\ u_7 \end{bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}}_{D_{f_0}^T} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_5 \\ u_7 \\ u_9 \end{bmatrix}}_{\mathbf{u}_f}$$

B. Suppose we choose node 3 as ground and set $u_3 = 0$ yielding

$$\underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_5 \\ u_7 \\ u_9 \end{bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{D_{f_0}^T} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_5 \\ u_7 \\ u_9 \end{bmatrix}}_{\mathbf{u}_f}$$

The *completely reduced solution* to the voltage-source KVLs is

$$\mathbf{u}_g = \mathbf{p}_g + Z\mathbf{u} \tag{18}$$

where the columns of $Z \in \mathbb{R}^{n_g \times n}$ are in the null space of A_{v_g} and the vector $\mathbf{u} \in \mathbb{R}^n$ stores the minimal list of independent variables needed to fully analyze the circuit. Because our goal is to state our circuit equations in the form of our desired equilibrium equation framework (1), we find it helpful to define matrices

$$A_r = A_{r_g} Z, \qquad A_{v_g} Z = 0, \qquad \text{and} \qquad A_i = A_{i_g} Z. \tag{19}$$

These matrices encode the reduction process by eliminating all reference to variables whose values can be calculated from our minimal set of unknown node voltages. Looking back at our reduced system (12), we substitute in our completely reduced solution (18) for $\mathbf{u}_g$ and then multiply the entire equation (12) on the left-hand side by the matrix Z^T. This yields the matrix equation

$$A_r^T G A_r \mathbf{u} = A_r^T G \mathbf{b} - \mathbf{f}, \tag{20}$$

where $\mathbf{b} = -A_{r_g}\mathbf{p}_g$ and $\mathbf{f} = A_i^T \mathbf{i}_i$. This equation is in the desired equilibrium framework (1). Moreover, for almost any circuit that is used in real-world applications, the *stiffness* matrix $K = A_r^T G A_r$ is nonsingular. For a novel proof of this fact, please contact the first author.

7 Step 7: Solve the equilibrium equation for the circuit

Assume we choose u_3 and u_5 as the independent node potentials for the two generalized nodes with node 9 selected as ground. This results in a linear-system of equations given as

$$\underbrace{\begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 2 & -2 & 0 & 0 \\ -1 & -2 & 8 & -1 & -2 \\ 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & -2 & -1 & 3 \end{bmatrix}}_{A_r^T\, G\, A_r} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_5 \\ u_7 \\ u_9 \end{bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} 6.0 \\ 5.0 \\ -20.0 \\ 7.5 \\ -5.0 \end{bmatrix}}_{A_r^T\, G\, \mathbf{b} - \mathbf{f}}$$

Solving this system and substituting the value of the vector $\mathbf{u}$ into our reduced solution to the voltage-source linear-systems problems (18), we produce a model for the electrical behavior of our example circuit, seen in column 2 of Table 3 below.

Node Variable	LANA model values (V)	MultiSim model values (V)	Physical measurement values (V)
u_1	2.6905	2.6905	2.680
u_2	−0.8095	−0.8095	−0.780
u_3	−3.3095	−3.3095	−3.272
u_4	−5.8095	−5.8095	−5.730
u_5	0.1190	0.1191	0.139
u_6	1.6905	1.6905	1.691
u_7	−3.8333	−3.8333	−3.766
u_8	−4.8810	−4.8810	−4.820
u_9	0.0000	0.0000	0.000

Table 3: Model verification for example circuit in Figure 1.

A rich feature of this modeling activity is that students can check their work by using an electronic circuit simulation program like MultiSim [34] or by prototyping their circuit and taking measurements with a digital multimeter [5]. In Table 3, columns 2 and 3 align very closely with each other indicating that the LANA algorithm replicates the results produced by commercially available CAD software programs. By engaging with this modeling activity, students get to see behind the curtain of CAD programs and understand some of the math behind that type of software. Column 4 of Table 3 is noticeably different than both columns 2 and 3 since the data in column 4 is produced by taking measurements using a digital multimeter on a physical breadboard circuit.

One major goal of the entire activity is to have students engage in authentic mathematical modeling including model verification. When we invite students to compare real-world data with modeled behavior predicted by mathematical algorithms, they have to do lots of deep thinking to decide for themselves if their math results are sufficiently accurate and valid to describe the physical phenomenon they are studying. In the process, students develop technical skills that are transferable to their future academic and career pursuits. It is amazing to see the transformation in attitude, interest, and excitement as students learn that they don't have to trust their teacher but instead can decide for themselves if the linear algebraic techniques they study in class are useful in applied contexts related to their majors.

Students can use the output to the LANA algorithm to solve very common problems in circuit analysis like finding the values of any circuit variable. The moment that the node voltage potential vector $\mathbf{u}_g$ is known, we can calculate the value of any other circuit variable. For example, we can use the element-specific KVLs (11) to find the voltage drop across the resistors and the current sources. We calculate the current flowing through any resistor in the circuit using the the matrix version of Ohm's law equation in conductance form (8). The currents running through the voltage sources can be found using our KCL equation

$$A_{v_g}^T \cdot \mathbf{i}_v = -A_{r_g}^T G A_{r_g} \mathbf{u}_g - A_{i_g}^T \mathbf{i}_i.$$

In this equation, the left-hand side includes our desired unknown variables while the right-hand side is completely determined by our previously stated constants. In other words, students who study this algorithm can solve a wide variety of problems from their introductory courses in circuit analysis.

CONCLUSION

The project described in this paper adds to a growing list of resources that engage linear algebra students in active learning on applied projects related to students' larger academic and career interests [2], [26], [27], [38]. These activities can be designed to address the four pillars of inquiry-based mathematics education [28] which are to deeply engage students in meaningful tasks, to have students collaborate, to inquire about and build on student ideas, and to foster equity in STEM classrooms. The design of this project also fits nicely into a framework for creating good interdisciplinary problems as suggested by Reinholtz et al. [42], who identify the following seven qualities as important to the creation *good* problems: (1) This project is accessible because it offers a low-floor problem that is easy for students to start. Students need only have access to a circuit example and can then begin implementing the steps of the LANA algorithm. (2) Circuit simulation is generalizable to harder tasks that relate to some students' future academic and career interests. For evidence supporting this claim, see the list of high-ceiling problems found in Appendix A. (3) The LANA algorithm also provides multiple solution paths in which students can compare and contrast their work to develop deeper knowledge. (4) In analyzing electric circuits using LANA, students get insights into core linear-algebraic concepts and practices like matrix-matrix multiplication as well as nonsingular and general linear-systems problems. (5) The mathematical techniques needed in the algorithm are typical elements in a first linear algebra course. (6) The entire project is context-rich since each circuit example is related to real-world engineering problems. (7) Finally, five years of teaching and learning experience with this project indicate that it is possible to use the LANA algorithm to create an enjoyable, playful, and interesting environment that motivates students to work on applied problems. Our experience with project-based, flipped-learning experiences has been positive, and for our students, powerful. We unreservedly recommend this type of work as vehicles for teaching and learning.

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APPENDIX A: OPEN-ENDED INQUIRY TASKS

1. Use the LANA algorithm along with the theory you learned in introductory linear algebra to find the values of the node voltage potentials. Verify your solutions via some other method. For example, you might:
   - A. Simulate the circuit using the free online MultiSim software [34].
   - B. Prototype the circuit [5] and measure with a digital multimeter.
2. Use output vector $\mathbf{u}_g$ from the LANA algorithm to:
   - A. Find all entries in the vectors $\mathbf{v}_r$, $\mathbf{v}_i$, $\mathbf{i}_r$, and $\mathbf{i}_v$.
   - B. Find the current vector $\mathbf{i}$ and analyze the flow of currents in your circuit. Then draw a directed graph model for the current flow in the entire network. What do you notice?

3. Make sense of and interpret the entries of all matrices used in this model:

   A. What do the entries of matrix $K = A_r^T G A_r$ from equation (20) represent?

   B. What do the vectors $\mathbf{f} = A_i^T \mathbf{i}_i$ and $\mathbf{b} = -A_{r_g} \mathbf{p}_g$ represent?

   C. Why is the process of melding all nodes within each generalized node equivalent to multiplying A_{r_g} on the right by Z_{v_g}?

   E. How is the matrix $A_r = A_{r_g} Z$ related to the deactivated resistor network? What information about the circuit can you read from A_r? What information is missing from this matrix?

4. Make sense of each entry in the various matrix equations from a physics or engineering perspective. How are the entries of these equations related to knowledge you are building in your other classes? In your response, address the following equations:

   A. KCL equations: $A_g^T \mathbf{i} = \mathbf{0}$.

   B. BCR equations: $\mathbf{i}_r = G\,\mathbf{v}_r$.

   C. KVL equations: $A_g\,\mathbf{u}_g = \mathbf{v}$.

5. Play with the LANA algorithm. Explore each step slowly and deliberately by answering the following questions:

   A. What happens if you switch the assigned reference directions for the resistors in the circuit? How does this change the matrix $K = A_r^T G A_r$ from equation (20)?

   B. Change the values of the power sources or resistors or both. Which parts of the equilibrium equation (20) change and which remain the same? Why?

   C. What happens if you switch your choice of free variables? How many different options do you have for this example circuit? How many options do you have for a general circuit? What do those options depend on?

   D. What happens if you switch your choice of ground node? How many options do you have?

   E. What if you want to ground a node that was not part of your chosen free variables? What changes do you need to make to do this?

   F. How many different nonsingular linear-systems problems can you generate using the LANA algorithm applied to a single circuit diagram? What are the relationships between the solutions to your various systems? Can you get one solution from the other? If so, how? If not, why not?

6. Design an example using circuits from other classes or get creative. Analyze your circuit using LANA and verify your output using your favorite circuit simulator or by prototyping your circuit and taking measurements.

7. How many different ways can you prove that the columns of Z_{v_g} form a basis for null(A_{v_g})? How many unique proofs can you create to show that the matrix $K = A_r^T G A_r$ from equation (20) is nonsingular?
8. How can you make the matrix $K = A_r^T G A_r$ from equation (20) singular? What features of a circuit would have to be present for the nonsingularity proof in LANA to fail? Design a circuit that produces a singular matrix K. How can you adapt your approach if you encounter such a circuit in the wild?
9. Write a MATLAB script file that produces the desired output(s) for any resistor network with DC voltage and current sources.
10. What happens if you were to eliminate the ground node constraint first and then eliminate the voltage source constraints? What adaptations can you make to the LANA algorithm to produce the end result by switching the order in which you eliminate variables?
11. How can you measure total system energy? What happens when you take the dot product between **v** and **i**? How is that output related to the conservation of energy? How is this related to Tellegen's Theorem [13, p. 1052 – 1055]?
12. How might you extend the LANA algorithm to deal with dependent power sources or to run AC analysis on RCL circuits? How are these extensions related to the modified nodal analysis [25] algorithm? How is the RCL modeling problem related to the eigenvalue problem used to model coupled mass-spring systems [2]? How does this inform your understanding of the electrical-mechanical correspondence [37, p. 320]?

Linear Algebraic Nodal Analysis: An Applied Project for a First Course in Linear Algebra

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Linear Algebraic Nodal Analysis: An Applied Project for a First Course in Linear Algebra

Abstract: Many students who enroll in a first course in linear algebra major in STEM disciplines other than mathematics. Teachers who serve such students may find it difficult to provide authenic problems from these broader areas that ignite students' interest in linear algebra. In this paper, we highlight an interdisciplinary learning activity that engages students in using linear systems of equations to model the behavior of practical electric circuits. This exercise fits nicely into standard introductory linear algebra curricula and is designed to excite students majoring in engineering, physics, or applied mathematics. We also include references to a collection of open-access resources to support instructors who want to use this material in project-based, flipped-learning, inquiry-oriented, or independent-study environments.

Keywords: Mathematical modeling, matrix multiplication, nonsingular linear systems, block matrix, electric circuit analysis, nodal analysis

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