Step-by-step guide to the linear-algebraic nodal analysis (LANA) algorithm

Jeffrey A. Anderson

INPUT:

Suppose we are given a complete description of an electric circuit containing only resistors, dc voltage sources, and dc current sources. The LANA algorithm proceeds with the steps provided below.

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1. Identify and label the entire set of nodes in our circuit.

   We begin our work by identifying the nodes in our circuit. To do so, we use the *node identification heuristic* in which we erase the element bodies of each ideal circuit element. The contiguous segments of conductor that remain are known as the *nodes* of the circuit. We label the nodes of our circuit using positive integers.

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2. Model the circuit as a directed graph.

   2A. Track the dimensions of key features in our circuit.

   Construct a digraph model $\mathcal{G}$ for our circuit by specifically the dimensions of key features. To do so, we define the following nonnegative integers:

$$\begin{aligned} n_g &= \text{the total number of nodes in the set } \mathcal{N} \text{ (including the ground node)},\\ m_r &= \text{the number of resistors in the circuit},\\ m_v &= \text{the number of dc voltage sources in the circuit},\\ m_i &= \text{the number of dc current sources in the circuit},\\ m &= \text{the total number of elements in our circuit.} \end{aligned}$$

   Each individual circuit element corresponds to a unique edge in our digraph and the set of edges $\mathcal{E}$ contains exactly m elements, with

$$m = m_r + m_v + m_i.$$

   2B. Orient and enumerate the edges of the digraph.

   Replace each circuit element with an edge in our digraph. If an edge models a current source, we orient this edge in the same direction as the flow of current in that source. We orient an edge that corresponds to a voltage source from the positive "+" lead to the negative "-" lead of the associated source. Finally, we assign arbitrary directions to all edges corresponding to the resistors in our circuit.

   We choose a special enumeration scheme for the edges of our digraph. First, we count and label all edges corresponding to resistors as edges $e_1, e_2, ..., e_{m_r}$. Next, we continue our count by labeling the edges corresponding to voltage sources, yielding edges $e_{m_r+1}, ..., e_{m_r+m_v}$. Finally, we enumerate our edges corresponding to current sources as $e_{m_r+m_v+1}, ..., e_m$. Each time we increment our edge index, we follow the same order determine by the enumeration scheme within each element type given in the original description of our circuit. For example, edge e_i encodes the reference current direction assigned to resistor i for $i = 1, 2, ..., m_r$.

   2C. Draw a directed graph model of the circuit.

3. Create the entire incidence matrix.

Form the entire incidence matrix $A_g \in \mathbb{R}^{m \times n_g}$ using the entry-by-entry definition

$$a_{jk} = \begin{cases} 1 & \text{if edge } e_j \text{ leaves node } k, \\ -1 & \text{if edge } e_j \text{ enters node } k, \\ 0 & \text{otherwise,} \end{cases}$$

for $j = 1, 2, ..., m$ and $k = 1, 2, ..., n_g$. The rows and columns of this matrix correspond to the edges and nodes of our digraph, respectively. We create a block-partition description of our entire incidence matrix A_g using our enumeration scheme for our digraph's edges, with

$$A_g = \begin{bmatrix} A_{r_g} \\ \hdashline A_{v_g} \\ \hdashline A_{i_g} \end{bmatrix}$$

where the subblocks of this matrix are given by

$$A_{r_g} \in \mathbb{R}^{m_r \times n_g}, \qquad A_{v_g} \in \mathbb{R}^{m_v \times n_g}, \qquad \text{and} \qquad A_{v_g} \in \mathbb{R}^{m_i \times n_g}.$$

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4. Create all circuit vectors.

4A. Define circuit variables for the node voltage potentials.

We create the *entire list of node voltage potentials* and store these variables in the vector $\mathbf{u}_g \in \mathbb{R}^{n_g}$. This list of node voltage potentials is organized as a column vector. At this point in the modeling process, we have yet to designate a ground node. Thus, the vector $\mathbf{u}_g$ contains all nodes identified in step 1 of this algorithm.

4B. Define circuit variables for the voltage drops across each element.

Define the *voltage-drop vector* $\mathbf{v} \in \mathbb{R}^m$ whose individual entries store the voltage drop across each circuit element, with

$$\mathbf{v} = \begin{bmatrix} \mathbf{v}_r \\ \hdashline \mathbf{v}_v \\ \hdashline \mathbf{v}_i \end{bmatrix} \qquad \text{where} \qquad \mathbf{v}_r \in \mathbb{R}^{m_r}, \ \mathbf{v}_v \in \mathbb{R}^{m_v}, \ \text{and } \mathbf{v}_i \in \mathbb{R}^{m_i}.$$

4C. Define circuit variables for the current running through each element.

Define the *current vector* $\mathbf{i} \in \mathbb{R}^m$ whose individual entries store the current running through each circuit element, with

$$\mathbf{i} = \begin{bmatrix} \mathbf{i}_r \\ \hdashline \mathbf{i}_v \\ \hdashline \mathbf{i}_i \end{bmatrix} \qquad \text{where} \qquad \mathbf{i}_r \in \mathbb{R}^{m_r}, \ \mathbf{i}_v, \in \mathbb{R}^{m_v}, \ \text{and } \mathbf{i}_i \in \mathbb{R}^{m_i}.$$

4D. Optional Extension to Step 4: Label all variables and reference directions.

Mark the original circuit schematic with all circuit variables and assigned reference directions. Define each reference current direction to run from the "+" lead to the negative "-" lead of the reference direction for the associated voltage drop variable and vice versa.

Notice that not all the scalar entries of the vectors $\mathbf{v}, \mathbf{i} \in \mathbb{R}^m$ are unknown variables. Specifically, the vectors $\mathbf{v}_v$ and $\mathbf{i}_i$ store known constants defined by the voltage and current levels provided by the independent sources we use to power our physical circuits. For $j = 1, ..., m_v$, we say that v_{v_j} represents the assigned voltage value of the jth dc voltage source Vj in our circuit, measured in volts. Using the same reasoning, for $j = 1, ..., m_i$, we say that i_{i_j} represents the assigned current running through the jth dc current source Ij in our circuit, measured in milliamps.

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5. State the entire set of circuit equations.

5A. State the *entire set of Kirchhoff's voltage laws in node-potential form.*

We recall that the complete set of circuit equations begins with Kirchhoff's voltage law (KVL) in node potential form. In the LANA algorithm, we state this set of equations using matrix multiplication:

$$A_g \, \mathbf{u}_g = \mathbf{v} \qquad \Leftrightarrow \qquad \begin{bmatrix} A_{r_g} \, \mathbf{u}_g \\ \hdashline A_{v_g} \, \mathbf{u}_g \\ \hdashline A_{i_g} \, \mathbf{u}_g \end{bmatrix} = \begin{bmatrix} \mathbf{v}_r \\ \hdashline \mathbf{v}_v \\ \hdashline \mathbf{v}_i \end{bmatrix}.$$

Using the subblock partition of the entire incidence matrix A_g and the voltage drop vector $\mathbf{v}$, we state the resulting element-specific KVLs as follows:

$$A_{r_g} \, \mathbf{u}_g = \mathbf{v}_r, \qquad A_{v_g} \, \mathbf{u}_g = \mathbf{v}_v, \qquad \text{and} \qquad A_{i_g} \, \mathbf{u}_g = \mathbf{v}_i.$$

5B. State the *brach constitutive relations* (BCRs) for the circuit.

To state the branch constitutive relations for the resistors in our circuit, we use a matrix version of Ohm's law given by the equation

$$\mathbf{v}_r = R \cdot \mathbf{i}_r \qquad \Leftrightarrow \qquad \mathbf{i}_r = G \cdot \mathbf{v}_r$$

where $R \in \mathbb{R}^{m_r \times m_r}$ is a diagonal matrix. Here, we store the appropriate resistance value in each diagonal entry, yielding scalar equations $v_k = r_k \cdot i_k$ for $k = 1, 2, ..., m_r$. Since the kth resistor Rk has positive resistance value $r_k > 0$, we can also rewrite our *Ohm's law equations in conductance form*, where the conductance matrix $G = R^{-1}$ is a diagonal and the main diagonal entries are defined by the individual conductances $g_k = 1/r_k$.

5C. State the *entire set of Kirchhoff's current laws.*

Finally we state the entire set of KCLs using the equation

$$A_g^T \, \mathbf{i} = \mathbf{0} \qquad \Leftrightarrow \qquad \left[\; A_{r_g}^T \; \middle| \; A_{v_g}^T \; \middle| \; A_{i_g}^T \; \right] \begin{bmatrix} \mathbf{i}_r \\ \hdashline \mathbf{i}_v \\ \hdashline \mathbf{i}_i \end{bmatrix} = \mathbf{0}.$$

Since both A_g and $\mathbf{i}$ are block partitioned matrices, we restate our *complete list of KCL's in sublock form* as

$$A_{r_g}^T \, \mathbf{i}_r + A_{v_g}^T \, \mathbf{i}_v + A_{i_g}^T \, \mathbf{i}_i = \mathbf{0}.$$

6. Determine all ordinary and generalized nodes in the circuit.

   Define a *generalized node* to be any set of nodes connected by a path of voltage sources. As we discuss below, for each generalized node we have only one independent node voltage potential. Any node to which no voltage source is connected is called an *ordinary node*.

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7. Ground the circuit.

   7A. Choose a *ground node* for the circuit.

   Determine a ground node, also called a *datum node*, $d \in \mathcal{N}$ and set $u_d = 0$. Set $n_0 = n_g - 1$ to be the *number of nonground nodes in the circuit*. Form the *ground node deflation matrix* $D_0 \in \mathbb{R}^{n_g \times n_0}$ by deleting the dth column of the $n_g \times n_g$ identity matrix. Deflate the corresponding row of $\mathbf{u}_g$ by setting $\mathbf{u}_0 = D_0^T \mathbf{u}_g$, where $\mathbf{u}_0 \in \mathbb{R}^{n_0}$ is called the *grounded voltage potential vector*.

   7B. Ground the circuit equations using deflation.

   Use the matrix D_0 from step 7A to form the grounded incidence matrices

$$A_0 = A_g\, D_0 = \begin{bmatrix} A_{r_0} \\ \hdashline A_{v_0} \\ \hdashline A_{i_0} \end{bmatrix}, \quad \text{with} \quad A_{r_0} = A_{r_g}\, D_0, \quad A_{v_0} = A_{v_g}\, D_0, \quad \text{and} \quad A_{i_0} = A_{i_g}\, D_0.$$

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8. State the *grounded circuit equations*.

   8A. State the *grounded KVLs in node potential form*.

   Once we ground the circuit, we can also ground the KVL equations by multiplying on the right by the ground node deflation matrix D_0, with

$$\mathbf{v} = A_0\, \mathbf{u}_0 = (A_g\, D_0)\,(D_0^T\, \mathbf{u}_g) \qquad \Leftrightarrow \qquad \begin{bmatrix} A_{r_0}\, \mathbf{u}_0 \\ \hdashline A_{v_0}\, \mathbf{u}_0 \\ \hdashline A_{i_0}\, \mathbf{u}_0 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_r \\ \hdashline \mathbf{v}_v \\ \hdashline \mathbf{v}_i \end{bmatrix}$$

   Once again, the subblock partition of the matrices A_g and $\mathbf{v}$ yield element-specific grounded KVLs

$$A_{r_0}\, \mathbf{u}_0 = \mathbf{v}_r, \qquad A_{v_0}\, \mathbf{u}_0 = \mathbf{v}_v, \qquad \text{and} \qquad A_{i_0}\, \mathbf{u}_0 = \mathbf{v}_i.$$

   8B. Combine the BCRs with the grounded KVLs.

   From our work in Step 5B of the LANA algorithm, we know that $\mathbf{i}_r = G \cdot \mathbf{v}_r$. We can combine this entire set of Ohm's law equations in conductance form with our grounded KVL equations for the resistor subblock of our circuit $A_{r_0}\, \mathbf{u}_0 = \mathbf{v}_r$ to see that

$$\mathbf{i}_r = G\, A_{r_0}\, \mathbf{u}_0.$$

   8C. State the *grounded KCLs*.

   We can also write the grounded version of the Kirchhoff's current law equations for our system with

$$A_0^T\, \mathbf{i} = \mathbf{0}.$$

   Since both A_0 and $\mathbf{i}$ are block partitioned matrices, we can restate our grounded KCLs in subblock form as

$$A_{r_0}^T\, \mathbf{i}_r + A_{v_0}^T\, \mathbf{i}_v + A_{i_0}^T\, \mathbf{i}_i = \mathbf{0}.$$

8D. Write the *modified nodal analysis matrix equation.*

As mentioned in the Step 4D of our algorithm, the entries of the vector $\mathbf{i}_i$ are known constants. Thus, we can bring the term $A_{i_0}^T \mathbf{i}_i$ in the subblock form of the grounded KCL equations to the right-hand side of the equals sign. This yields an equivalent system of grounded KCLs where the contributions of the dc current sources are now expressed on the right-hand side, with

$$A_{r_0}^T G A_{r_0} \mathbf{u}_0 + A_{v_0}^T \mathbf{i}_v = -A_{i_0}^T \mathbf{i}_i.$$

In this equation, we've replaced the vector $\mathbf{i}_r$ with it's equivalent node potential representation from step 8B of our algorithm. This is reminiscent of the well-known approach we take to discuss forced harmonic oscillators in which we write the forcing function on the right-hand side of our governing equations. If we combine these KCLs with the grounded KVLs for the voltage sources, we produce the modified nodal analysis equation

$$\begin{bmatrix} A_{r_0}^T G A_{r_0} & A_{v_0}^T \\ A_{v_0} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_0 \\ \mathbf{i}_\mathbf{v} \end{bmatrix} = \begin{bmatrix} -A_{i_0}^T \mathbf{i}_i \\ \mathbf{v}_v \end{bmatrix}.$$

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9. Identify the essential nodes, nonessential nodes, and supernodes.

i. If the chosen ground node lies within a generalized node, all other nodes within that generalized node are called *nonessential nodes* and they carry known node voltage potential values that can be calculated using specific entries of the vector $\mathbf{v}_v$. Find these values by manipulating the appropriate row(s) of the *entire set of KVLs for the voltage sources*

$$A_{v_g} \mathbf{u}_g = \mathbf{v}_v.$$

ii. Any generalized node that does not contain the ground node is referred to as a *supernode.*

ii. Ordinary nonground nodes are called *essential nodes.*

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10. Eliminate the node dependencies that arise due to the voltage sources.

10A. Determine the independent and dependent node voltage potentials.

i. For each essential node, assign one independent node voltage potential.

ii. For each supernode, assign exactly one independent node voltage potential. Once we've determined the independent node potential within each supernode, all other node voltage potentials within that supernode are now dependent. In particular, by algebraic manipulations of the appropriate row(s) of the equation $A_{v_g} \mathbf{u}_g = \mathbf{v}_v$, we can write the dependent node potentials in that super node in terms of our chosen independent node voltage potential and known voltage source values. Notice that we can eliminate one node voltage potential variable for each voltage source in our circuit.

iii. Determine the node voltage potential value of each nonessential node. This step may take some thought if we've grounded a generalized node that contains multiple voltage sources.

10B. Create a general solution to the entire set of voltage-source KVLs.

Let $n = n_0 - m_v$ be the number of independent node voltage potentials. Rewrite the vector $\mathbf{u}_g$ in the form

$$\mathbf{u}_g = \mathbf{p}_g + D_{v_g} \mathbf{u},$$

where $\mathbf{u} \in \mathbb{R}^n$ contains a chosen set of linearly independent node potentials, the columns of $D_{v_g} \in \mathbb{R}^{n_0 \times n}$ are in the Nul $\left(A_{v_g}\right)$ and the vector $\mathbf{p}_g \in \mathbb{R}^{n_g}$ is a particular solution to the general linear-systems problem $A_{v_g} \mathbf{u}_g = \mathbf{v}_v$.

10C. Create a general solution to the grounded set of voltage-source KVL equations.

Use the ground node deflation matrix D_0 to ground our general solution with

$$\mathbf{u}_0 = D_0^T \mathbf{u}_g = D_0^T \left(\mathbf{p}_g + D_v \mathbf{u}\right) = \mathbf{p}_0 + D\mathbf{u}$$

where the $n_0 \times 1$ vector $\mathbf{p}_0 = D_0^T \mathbf{p}_g$ and the *grounded voltage source deflation matrix* $D = D_0^T D_{v_g} \in \mathbb{R}^{n_0 \times n}$.

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11. State the maximally deflated circuit equations.

11A. Maximally deflate the incidence matrices.

Use the grounded voltage source deflation matrix $D = D_0^T D_{v_g}$ to form the *fully deflated incidence matrices*

$$A = A_0 D, \qquad \text{with} \qquad A_r = A_{r_0} D, \qquad A_v = A_{v_0} D, \qquad \text{and} \qquad A_i = A_{i_0} D.$$

Notice that by construction, we have the $A_v = 0 \in \mathbb{R}^{n_0 \times n}$. In fact, the columns of D form a basis for $\operatorname{Nul}(A_{v_0})$.

11B. Create the *grounded and deactivated nodal equations in nonsingular form.*

Multiply the left-hand side of the ground KCLs in subblock form from Step 8D by the matrix D^T and substitute the expanded version of vector $\mathbf{u}_0 = \mathbf{p}_0 + D\mathbf{u}$ into this equation to produce

$$A_r^T G A_{r_0} \left(\mathbf{p}_0 + D\mathbf{u}\right) = -A_i^T \mathbf{i}_i.$$

If we set $\mathbf{f} = A_i^T \mathbf{i}_i$ and $\mathbf{b} = -A_{r_0} \mathbf{p}_0$, we produce the system

$$K \mathbf{u} = A_r^T G \mathbf{b} - \mathbf{f},$$

where $K = A_r^T G A_r \in \mathbb{R}^{n \times n}$ and $\mathbf{f} \in \mathbb{R}^n$. With very reasonable assumptions on our original circuit, the matrix K is always nonsingular and positive definite. This matrix structure is popular in STEM modeling contexts and has received much attention over decades of research and development in numerical linear algebra. Moreover, this equation fits into a larger equilibrium equation framework espoused by Gilbert Strang.

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12. Solve the linear-algebraic nodal analysis equation.

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13. Completely describe the electrical behavior of the circuit.

13A. Calculate all the node voltage potentials.

13B. Calculate any desired circuit variable values.

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14. Verify by measuring the node voltage potentials on the real circuit.

OUTPUT:

This algorithm outputs the modeled values for all independent node voltage potentials. We can then use these node potential to solve for any circuit variable we desire.